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CALCULUS

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PREFACE TO THE THIRD EDITION

In this second revision a considerable part of the text has been rewritten. Many new exercises have been added, some of the exercises of the former edition have been omitted, and others have been changed. The general plan of the book, however, remains as it was.

The authors wish to acknowledge the helpful suggestions and constructive criticisms made by members of the Departments of Mathematics of the University of Wisconsin and of the Drexel Institute of Technology.

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MADISON, WIS.,
PHILADELPHIA, PA.,
November, 1937.

PREFACE TO THE FIRST EDITION

One of the purposes of the elementary working courses in mathematics of the freshman and sophomore years is to exhibit the bond that unites the experimental sciences. "The bond of union among the physical sciences is the mathematical spirit and the mathematical method which pervade them." For this reason, the applications of mathematics, not to artificial problems, but to the more elementary of the classical problems of natural science, find a place in every working course in mathematics. This presents probably the most difficult task of the textbook writer—namely, to make clear to the student that mathematics has to do with the laws of actual phenomena, without at the same time undertaking to teach technology, or attempting to build upon ideas which the student does not possess. It is easy enough to give examples of the application of the processes of mathematics to scientific problems; it is more difficult to exhibit by these problems how, in mathematics, the very language and methods of thought fit naturally into the expression and derivation of scientific laws and of natural concepts.

It is in this spirit that the authors have endeavored to develop the fundamental processes of the calculus which play so important a part in the physical sciences; namely, to place the emphasis upon the mode of thought in the hope that, even though the student may forget the details of the subject, he will continue to apply these fundamental modes of thinking in his later scientific or technical career. It is with this purpose in mind that problems in geometry, physics, and mechanics have been freely used. The problems chosen will be readily comprehended by students ordinarily taking the first course in the calculus.

A second purpose in an elementary working course in mathematics is to secure facility in using the rules of operation which must be applied in calculations. Of necessity large numbers of

drill problems have been inserted to furnish practice in using the rules. It is hoped that the solution of these problems will be regarded by teacher and student as a necessary part but not the vital part of the course.

While the needs of technical students have been particularly in the minds of the authors, it is believed that the book is equally adapted to the needs of any other student pursuing a first course in calculus. The authors do not believe that the purposes of courses in elementary mathematics for technical students and for students of pure science differ materially. Either of these classes of students gains in mathematical power from the type of study that is often assumed to be fitted for the other class.

In agreement with many others, the book is not divided into two parts, Differential Calculus and Integral Calculus. Integration with the determination of the constant of integration, and the definite integral as the limit of a sum, are given immediately following the differentiation of algebraic functions and before the differentiation of the transcendental functions. With this arrangement many of the most important applications of the calculus occur early in the course and constantly recur. Further, with this arrangement, the student is enabled to pursue more advantageously courses in physics and mechanics simultaneously with the calculus.

The attempt has been made to give infinitesimals their proper importance. In this connection Duhamel's Theorem is used as a valuable working principle, though the refinements of statement upon which a rigorous proof can be based have not been given.

The subjects of center of gravity and moments of inertia have been treated somewhat more fully than is usual. They are particularly valuable in emphasizing the concept of the definite integral as the limit of a sum and as a mode of calculating the mean value of a function. Sufficient solid analytic geometry is given to enable students without previous knowledge of this subject to work the problems involving solids. In the last chapter simple types of differential equations are taken up.

The book is designed for a course of four hours a week throughout the college year. But it is easy to adapt it to a three-hour course by suitable omissions.

The authors are indebted to numerous current textbooks for many of the exercises. To prevent distracting the student's attention from the principles involved, exercises requiring complicated reductions have been avoided as far as possible.

The book in a preliminary form has been used for two years with students in the College of Engineering of the University of Wisconsin. Many improvements have been suggested by our colleagues, Professor H. T. Burgess, Messrs. E. Taylor, T. C. Fry, J. A. Nyberg, and R. Keffer. Particular acknowledgment is due to the editor of this series, Professor C. S. Slichter, for suggestions as to the plan of the book and for suggestive criticism of the manuscript at all stages of its preparation.

The authors will feel repaid if a little has been accomplished toward presenting the calculus in such a way that it will appeal to the average student rather as a means of studying scientific problems than as a collection of proofs and formulas.

HERMAN W. MARCH,
HENRY C. WOLFF.

UNIVERSITY OF WISCONSIN,
November 6, 1916.

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CALCULUS

INTRODUCTION

1. Constant. Variable. Function. 1. A symbol of number or quantity to which a fixed value is assigned throughout the same problem or discussion is called a *constant*.

2. A symbol of number or quantity to which a succession of values is assigned in the same problem or discussion is called a *variable*.

Example. The mass or weight of mercury in a thermometer is constant. The number that results from measuring this quantity (weight) is a constant.

The volume of the mercury in the thermometer is variable. The number that results from measuring this quantity (volume) is a variable.

3. The variable y is said to be a *function* of the variable x if, when x is given, one or more values of y are determined.

4. The variable x, to which values are assigned at will, is called the *independent variable*, or the *argument of the function*.

5. The variable y, whose values are thereby determined, is called the *dependent variable*.

6. A variable y is said to be a function of several variables $u, v, w, \ldots$ if, when $u, v, w, \ldots$ are given, one or more values of y are determined.

7. The variables $u, v, w, \ldots$, to which values are assigned at will, are called the *independent variables*, or the *arguments of the function*.

Functions of a single variable or argument are represented by symbols such as the following: $f(x)$, $F(x)$, $\phi(x)$, $\psi(x)$. Functions of several arguments are represented by symbols such as $f(u, v, w)$, $F(u, v, w)$, $\phi(u, v, w)$.

2. The Power Function. 8. The function x^n, where n is a constant, is called the *power function.*

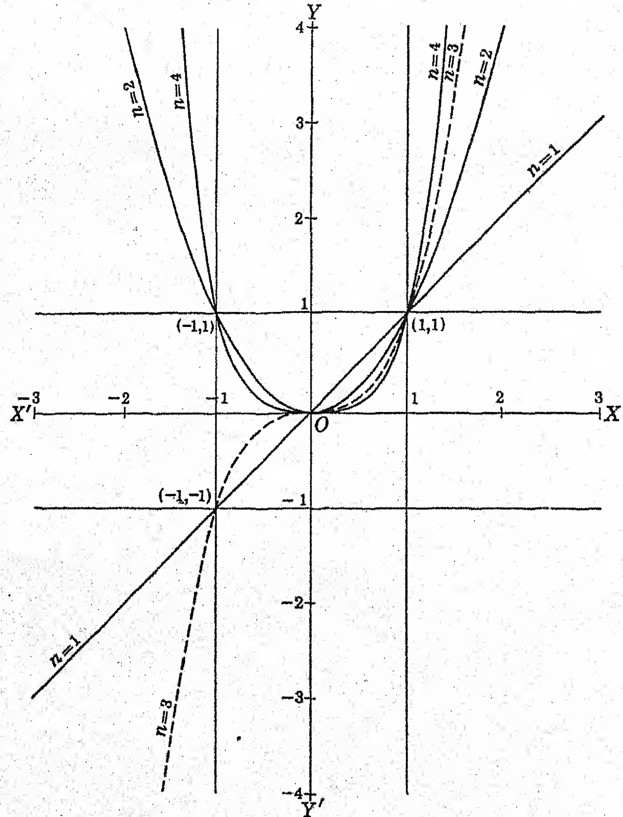

FIG. 1.—Curves for $y = x^n$, $n = 1, 2, 3$, and 4.

If n is positive the function is said to be of the *parabolic type,* and the curve representing such a function is also said to be of the parabolic type. If $n = 2$, the curve, $y = x^2$, is a *parabola.*

If n is negative, the function x^n is said to be of the *hyperbolic type,* and the curve representing such a function is also said to be

of the hyperbolic type. If $n = -1$, the curve, $y = x^{-1}$, is *an equilateral hyperbola.*

In Figs. 1, 2, 3, and 4, curves representing $y = x^n$ for different values of n are drawn. In Fig. 1, n has positive integral values; in Fig. 2, positive fractional values; in Fig. 3, negative integral values; and in Fig. 4, negative fractional values. The curves for

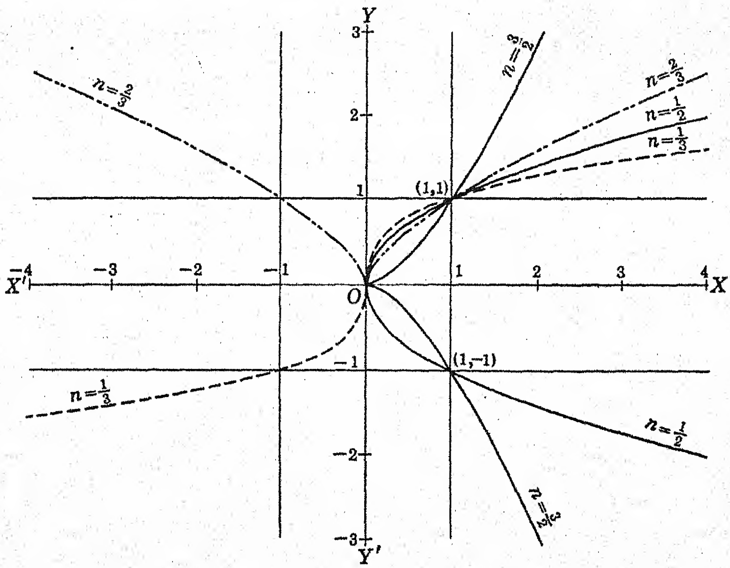

FIG. 2.—Curves for $y = x^n$, $n = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, and $\frac{3}{2}$.

$y = x^n$ all pass through the point (1, 1). They also pass through the point (0, 0) if n is positive. If n is negative, they do not pass through (0, 0). In the latter case the coordinate axes are asymptotes to the curves.

3. The Law of the Power Function. 9. In any power function, if x changes by a fixed multiple, y also changes by a fixed multiple.

The same law can be stated as follows:

10. In any power function, if x increases by a fixed percentage, y also increases by a fixed percentage.

The preceding statements are also equivalent to the following:

11. In any power function, if x runs over the terms of a geometrical progression, then y also runs over the terms of a geometrical progression.

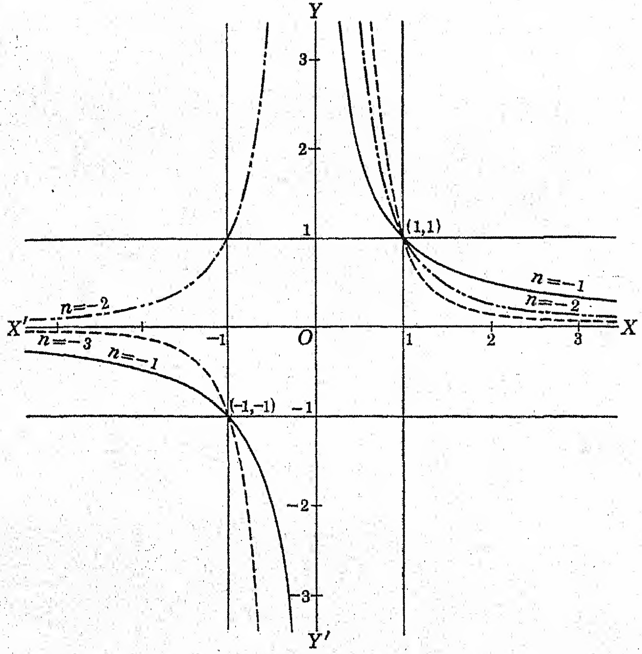

FIG. 3.—Curves for $y = x^n$, $n = -1, -2,$ and -3.

4. Polynomials. Algebraic Function. 12. A polynomial in x is a sum of a finite number of terms of the form ax^n, where a is a constant and n is a positive integer or zero. For example:

$$ax^3 + bx^2 + cx + d.$$

13. A polynomial in x and y is a sum of a finite number of terms of the form ax^my^n, where a is a constant and m and n are positive integers or zero. For example:

$$ax^2y^3 + bxy^2 + cx^2 + dy + e.$$

14. Functions of a variable x which are expressed by means of a finite number of terms involving only constant integral and

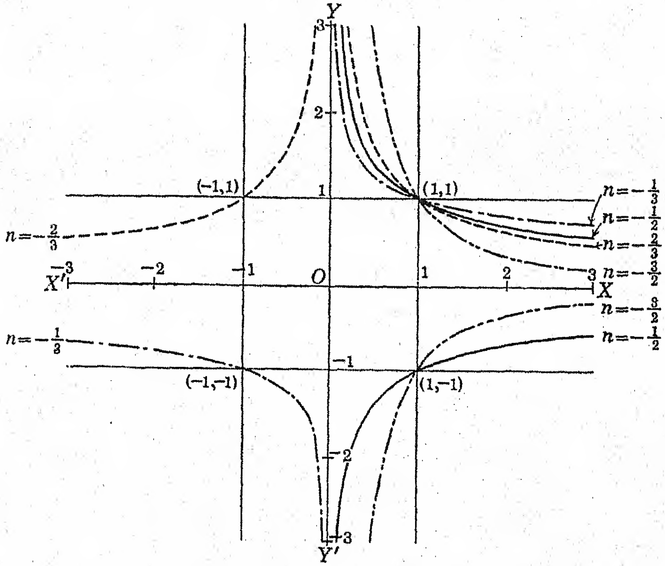

FIG. 4.—Curves for $y = x^n$, $n = -\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}$, and $-\frac{3}{2}$.

fractional powers of x and of polynomials in x are included in the class of functions known as *algebraic functions*[1] of x. For example:

(*a*) x^2. (*d*) $\dfrac{3}{x^2} + \dfrac{7}{x} + 1$.

(*b*) $x^{\frac{1}{3}} + (2x - 3)^{\frac{1}{4}}$. (*e*) $x + 5 + \dfrac{1}{\sqrt{x - 7}}$.

(*c*) $\sqrt{x^2 + 4x + 7} + 4x + 5$. (*f*) $\dfrac{3x^2 + 5x + 7}{x^3 - 3x + 2}$.

[1] A function of x defined by the equation $F(x, y) = 0$, where $F(x, y)$ is a polynomial in x and y, is an *algebraic function* of x. For example, $y = \sqrt{x^2 + 2}$ is an algebraic function of x. For by squaring and transposing, we obtain

$$y^2 - x^2 - 2 = 0,$$

in which the first member is a polynomial in x and y.

15. An algebraic function of x is said to be *rational* if it can be expressed by means of only integral powers of x together with constants.

Rational algebraic functions are divided into two classes: rational integral functions and rational fractional functions.

16. A *rational integral* function of x is a polynomial in x.

17. A rational fractional function is a quotient of two polynomials in x.

It is usually desirable to reduce rational fractional functions of x to a form in which the numerator is of lower degree than

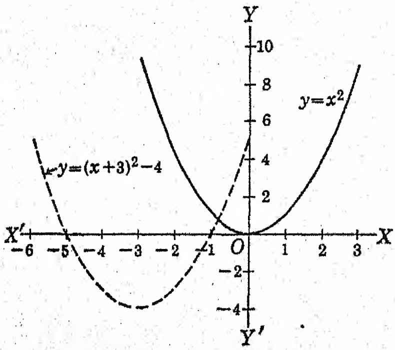

Fig. 5.

the denominator. This can always be done by performing long division.

Thus $y = \dfrac{x+3}{x+1}$ is equivalent to $y = 1 + \dfrac{2}{x+1}$, and $y = \dfrac{3x^2+5x+7}{x^2-3x+2}$ is equivalent to $y = 3 + \dfrac{14x+1}{x^2-3x+2}$.

5. Transcendental Functions. The circular (or trigonometric), the logarithmic, and the exponential functions are included in the class of functions known as *transcendental*[1] *functions.*

6. Translation. If, in the equation of a curve

$$f(x, y) = 0,$$

[1] All functions which are not algebraic functions as defined by the footnote on p. 5 are *transcendental functions.*

x is replaced by $(x - \alpha)$, the resulting equation,

$$f(x - \alpha, y) = 0,$$

represents the first curve translated parallel to the axis of x a distance α; to the right if α is positive; to the left if α is negative.

If y is replaced by $(y - \beta)$, the resulting equation,

$$f(x, y - \beta) = 0,$$

represents the original curve translated parallel to the axis of y a distance β; up if β is positive; down if β is negative. Thus $y = (x + 3)^2 - 4$ is the parabola $y = x^2$ translated three units to the left and four units down (see Fig. 5).

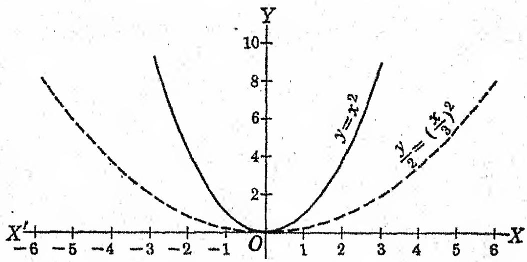

FIG. 6.

7. Elongation or Contraction, or Orthographic Projection, of a Locus. The substitution of $\frac{x}{a}$ for x in the equation of any locus multiplies all of the abscissas by a.

This transformation may be considered as the orthographic projection of a curve lying in one plane upon another plane, the two planes intersecting in the axis of y. If $a < 1$, the second curve is the projection of the former curve upon a second plane through the Y-axis and making an angle α, whose cosine is equal to a, with the first plane. If $a > 1$, the first curve is the projection of the second when the cosine of the angle between their planes is $\frac{1}{a}$.

Similarly, the substitution of $\frac{y}{a}$ for y in the equation of a locus multiplies the ordinates by a. The interpretation from the

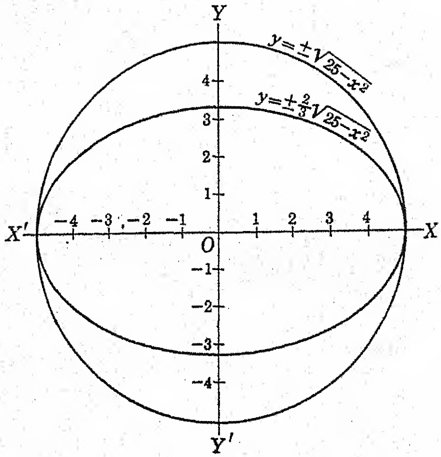

FIG. 7.

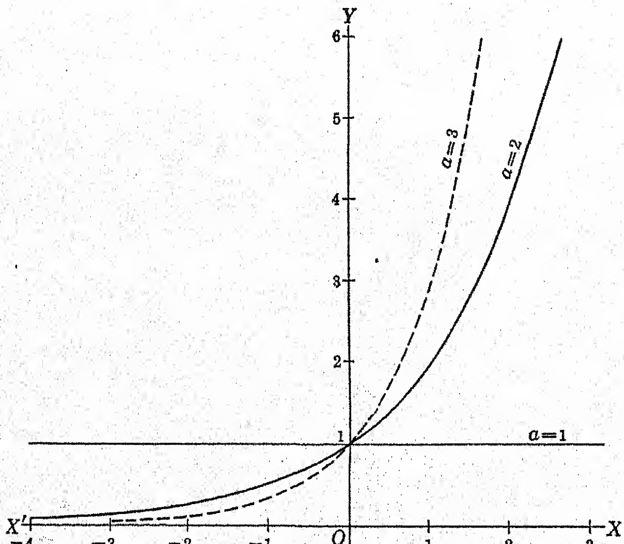

FIG. 8.—Curves for $y = a^x$, $a = 1, 2$, and 3.

standpoint of orthographic projection is evident from what has just been said (see Figs. 6 and 7).

8. Shear. The curve $y = f(x) + mx$ is the curve $y = f(x)$ sheared in the line $y = mx$ in such a way that the y-intercepts remain unchanged. Every point on the curve $y = f(x)$ to the right of the Y-axis is moved up (down if m is negative) a distance proportional to its abscissa; and every point to the left of the Y-axis is moved down (up if m is negative) a distance proportional to its abscissa. The factor of proportionality is m.

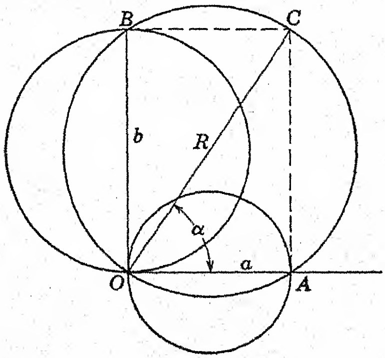

FIG. 9.

9. The Function a^x. In Fig. 8 are given the graphs of $y = a^x$, for the values $a = 1$, 2, and 3. By reflecting these curves in the line $y = x$ we have the corresponding curves for $y = \log_a x$.

The exponential function $y = a^x$ has the property that if x is given a series of values in arithmetical progression the corresponding values of y are in geometrical progression.

10. The Function sin x. The function $y = \sin x$ is a periodic function of period 2π. The function $y = \sin mx$, whose graph is readily obtained from that of $y = \sin x$, has the period $\frac{2\pi}{m}$. Similar statements can be made concerning the functions $y = \cos x$ and $y = \cos mx$.

11. The Functions $\rho = a \cos \theta$, $\rho = b \sin \theta$, and $\rho = a \cos \theta + b \sin \theta$. The function $\rho = a \cos \theta$ is the circle OA, Fig. 9, and $\rho =$

$b \sin \theta$ is the circle OB, Fig. 9. The function $\rho = a \cos \theta + b \sin \theta$ can be put in the form $\rho = R \cos (\theta - \alpha)$ where $R = \sqrt{a^2 + b^2}$. and where $\cos \alpha = \frac{a}{R}$ and $\sin \alpha = \frac{b}{R}$. This function is represented by a circle, Fig. 9, passing through the pole, with diameter equal to R, and with the angle AOC equal to α. The maximum value of the function is R and the minimum value is $-R$.

12. Fundamental Transformations of Functions. It is valuable to formulate the transformations of simple functions, that most commonly occur, in terms of the effect that these transformations have upon the graphs of the functions. The following list of theorems on loci contains useful facts concerning these transformations:

THEOREMS ON LOCI

I. If x be replaced by $(-x)$ in any equation containing x and y, the new graph is the reflection of the former graph in the Y-axis.

II. If y be replaced by $(-y)$ in any equation containing x and y, the new graph is the reflection of the former graph in the X-axis.

III. If x and y be interchanged in any equation containing x and y, the new graph is the reflection of the former graph in the line $y = x$.

IV. Substituting $\left(\frac{x}{a}\right)$ for x in the equation of any locus multiplies all abscissas by a.

V. Substituting $\left(\frac{y}{b}\right)$ for y in the equation of any locus multiplies all ordinates of the curve by b.

VI. If $(x - a)$ be substituted for x throughout any equation, the locus is translated a distance a in the x-direction.

VII. If $(y - b)$ be substituted for y in any equation, the locus is translated the distance b in the y-direction.

VIII. The addition of the term mx to the right side of $y = f(x)$ shears the locus $y = f(x)$ in the line $y = mx$.

IX. If $(\theta - \alpha)$ be substituted for θ throughout the polar equation of any locus, the curve is rotated about the pole through the angle α.

X. If the equation of any locus is given in rectangular coordinates, the curve is rotated through the positive angle α by the substitutions

$$x \cos \alpha + y \sin \alpha \quad \text{for } x$$

and

$$y \cos \alpha - x \sin \alpha \quad \text{for } y.$$

CHAPTER I

DERIVATIVE

In elementary analysis the student investigated the dependence of a function upon one or more variables with the help of algebra and geometry.

He is now to study a very powerful method of investigating the behavior of functions, the method of *the infinitesimal calculus*, which was discovered independently by Newton and Leibnitz

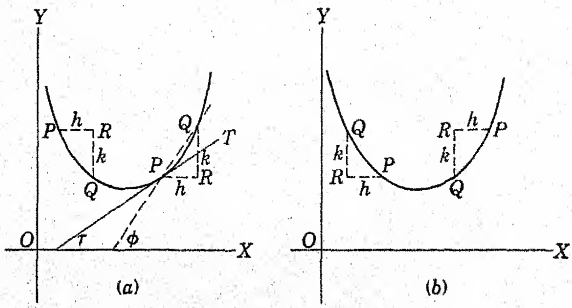

FIG. 10.

in the latter part of the seventeenth century. This method has made possible the great development of mathematical analysis and of its applications to problems in almost every field of science, particularly in engineering and physics.

13. Increments. Slope of Tangent. Let $y = f(x)$ express a functional relation existing between the variables x and y, and let the curve of Fig. 10 represent this relation graphically. Let x_1 and y_1, the coordinates of the point P, be corresponding values of x and y. If the independent variable x changes from x_1 to $x_1 + h$, the dependent variable y will change from y_1 to a certain value $y_1 + k$. Let the point Q on the curve have the coordinates $x_1 + h$ and $y_1 + k$.

Draw the line PQ through the points P and Q and the line PT tangent to the curve at the point P. Let ϕ and τ be, respectively, the inclinations of the secant and tangent lines. Since, by definition, the tangent line is the limiting position of the secant line as Q approaches P, the angle ϕ approaches the angle τ.

From the figure we see that $\frac{k}{h} = \tan \phi$ is the slope of the secant line PQ. Consequently

$$\lim_{h \to 0} \frac{k}{h} = \lim_{h \to 0} \tan \phi = \tan \tau,$$

the slope of the tangent line.[1]

The quantity h, the change in the variable x, is called the increment of x, and the quantity k, the change in the variable y, is called the increment of y. The increment of x may be positive or it may be negative as shown in the figure; in (a) it is positive and in (b) it is negative. Even though h, the change in x, is negative, it is still called an increment. Likewise k, the increment of y, may be positive or it may be negative.

Since the comparison of increments of related variables is fundamental in the study of the calculus, it is desirable to adopt a symbol for the words "increment of." The symbol used is the Greek capital letter "delta," placed before the variable. Thus Δx (read "delta x") means an increment of x. It corresponds to h used above. Similarly Δy, Δs, Δt, etc. (read "delta y," "delta s," "delta t," etc.) represent increments of the variables y, s, t, etc., respectively. In this notation the discussion previously given of the slope of a line tangent to a curve is written:

$$\frac{\Delta y}{\Delta x} = \tan \phi, \text{ the slope of the secant line } PQ,$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \tan \phi = \tan \tau, \text{ the slope of the tangent line } PT.$$

14. The Function y = x². As an illustration of the discussion of the preceding section, let us consider the functional relation $y = x^2$, whose graph is shown in Fig. 11.

[1] The symbol $\lim_{h \to 0} \frac{k}{h}$ is read "the limit of $\frac{k}{h}$ as h approaches zero."

Let P: (x_1, y_1) be a point on the curve, so that

$$y_1 = x_1^2. \tag{1}$$

Let x_1 take on an increment Δx. Then y_1 will take on an increment Δy, and we have

$$y_1 + \Delta y = (x_1 + \Delta x)^2. \tag{2}$$

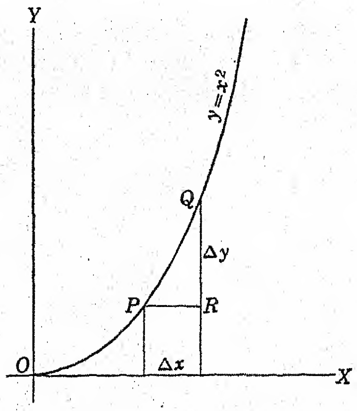

Fig. 11.

From (1) and (2),

$$\Delta y = 2x_1\Delta x + (\Delta x)^2.$$

Then

$$\frac{\Delta y}{\Delta x} = 2x_1 + \Delta x, \tag{3}$$

and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x_1. \tag{4}$$

From (4) we see that the slope of the line tangent to the curve $y = x^2$ at the point (x_1, y_1) is equal to twice the abscissa of that point. Thus the slope of the tangent at (3, 9) is 6; at (−1, 1) is −2; at $(\frac{1}{2}, \frac{1}{4})$ is 1; and so on. Subscripts were used above to indicate that a definite point on the curve was under consideration. Hereafter the subscripts will be omitted, since it is clear that the argument holds for any point on the curve.

From equation (3) it is seen that the ratio of Δy to Δx is a function of both x and Δx, *i.e.*, its value depends upon the position of the point P on the curve and upon the magnitude and sense of Δx.

It will be interesting to see how the ratio $\frac{\Delta y}{\Delta x}$ approaches its limit at a definite point on the curve. By using equation (3) this ratio was calculated at the point (0.2, 0.04) for successively smaller and smaller values of Δx. The results, which the student should verify, are given in the adjoining table. We observe that as Δx is taken smaller and smaller, the ratio $\frac{\Delta y}{\Delta x}$ approaches more and more closely a value in the vicinity of 0.4. Equation (4) shows that the limit is exactly 0.4.

Δx	Δy	$\frac{\Delta y}{\Delta x}$
0.4	0.32	0.8
0.2	0.12	0.6
0.1	0.05	0.5
0.05	0.0225	0.45
0.02	0.0084	0.42
0.01	0.0041	0.41
0.005	0.002025	0.405
0.002	0.000804	0.402
0.001	0.000401	0.401

15. The Function $y = x^3$. Let us consider the function

$$y = x^3. \tag{1}$$

Let x take on an increment Δx. Then

$$y + \Delta y = (x + \Delta x)^3$$

and

$$\begin{aligned}\Delta y &= (x + \Delta x)^3 - x^3 \\ &= 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3.\end{aligned}$$

Then

$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x(\Delta x) + (\Delta x)^2. \tag{2}$$

Using equation (2), the student will construct, for the point (0.4, 0.064) on the curve, a table similar to that of the preceding section. From this table it would appear that the ratio of Δy to Δx approaches nearer and nearer the value 0.48 as Δx approaches zero.

From equation (2) we see that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 3x^2. \tag{3}$$

Equation (3) shows that the limit of the ratio of Δy to Δx is exactly 0.48 when $x = 0.4$.

Exercises

Construct a table similar to the table given in §14 for each of the functions 1, 2, 3, and 4 for the point whose abscissa is 0.02.

1. $y = 2x^2$.

2. $y = 3x^2$.

3. $y = 3x^3$.

4. $y = x^4$.

5. Find $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ for each function of Exercises 1, 2, 3, and 4.

16. Velocity. Let a point move along a straight line, and at any time t let the position of the point, as measured from a certain fixed point O on the line, be denoted by s. Here s may be positive or negative, depending upon the position of the point and upon the convention adopted as to the positive sense of s. It is clear that s is a function of t. The functional relation between s and t can be expressed analytically by the equation $s = f(t)$ or it can be represented graphically by a curve such as that of Fig. 12.

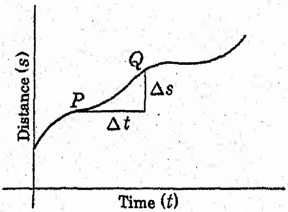

FIG. 12.

At a certain instant t, the moving point is at the position s, while at a later instant $t + \Delta t$, it is at the position $s + \Delta s$, the point having moved through a distance Δs in the time Δt. The ratio $\frac{\Delta s}{\Delta t}$ is defined as the *average velocity* of the point for the interval of time Δt. As Δt is taken smaller and smaller, the corresponding values of $\frac{\Delta s}{\Delta t}$ represent the average velocities for smaller and smaller intervals of time. The limit of $\frac{\Delta s}{\Delta t}$ as Δt approaches zero is defined as *the velocity of the point* at the instant t.

17. Velocity of a Falling Body. As an illustration of the application of the method of increments, let us find the velocity of a falling body. The law of motion has been experimentally determined to be

$$s = \tfrac{1}{2}gt^2,$$

where s is the distance through which the body falls from rest in time t. If s is measured in feet and t in seconds, g is 32.2 approximately.

The relation connecting s and t is represented graphically by the curve in Fig. 13. If t takes on an increment Δt, s takes on an increment Δs. These increments are represented in the figure by PR and RQ, respectively.

Since $s = \frac{1}{2}gt^2$,

$$s + \Delta s = \tfrac{1}{2}g(t + \Delta t)^2. \quad (1)$$

Hence

$$\Delta s = \tfrac{1}{2}g(t + \Delta t)^2 - \tfrac{1}{2}gt^2,$$

or

$$\Delta s = gt\Delta t + \tfrac{1}{2}g(\Delta t)^2. \quad (2)$$

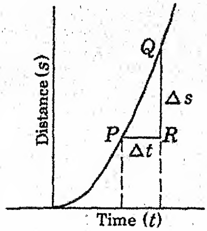

FIG. 13.

This is the distance through which the body falls in the interval Δt counted from the time t. The quotient $\dfrac{\Delta s}{\Delta t}$ is the average velocity for the interval Δt. The velocity at t has been defined as $\lim\limits_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t}$. To find this limit, divide (2) by Δt and obtain

$$\frac{\Delta s}{\Delta t} = gt + \tfrac{1}{2}g\Delta t,$$

the average velocity for the interval Δt, from which

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = gt, \quad (3)$$

the velocity at t. Thus the velocity at the end of 3 seconds is 96.6 feet per second; at the end of 4 seconds, 128.8 feet per second.

18. Derivative. The limit of the ratio of the increment of a function of an independent variable to the increment of the independent variable, as the latter increment approaches zero, is very useful in studying the behavior of the function. This limit is called *the derivative of the function with respect to the variable.* Hence the following definition:

The derivative of a function of a single independent variable with respect to that variable is the limit of the ratio of the increment

of the function to the increment of the variable as the latter increment approaches zero. The derivative of a function y with respect to a variable x is denoted by the symbol $\frac{dy}{dx}$. This symbol is read, *the derivative of y with respect to x.* The symbols $\frac{dy}{dt}$, $\frac{ds}{dy}$, $\frac{dt}{dy}$, etc., are read "the derivative of y with respect to t, the derivative of s with respect to y, the derivative of t with respect to y," etc. *The process of finding the derivative is called differentiation.*

If $y = f(x)$, the derivative of this function with respect to x is often denoted by the symbol[1] $f'(x)$. Thus, if $f(x) = x^2$,

$$\frac{dy}{dx} = f'(x) = 2x. \text{ (See §14.)}$$

In this chapter two important interpretations of the derivative have been pointed out, one of them associated with geometry, the other with physics. Thus from §13 it follows that the derivative of y with respect to x is equal to the slope of the line tangent to the curve representing y as a function of x. From §16 it follows that the velocity of a point moving in a straight line is equal to the derivative of s with respect to t, where s is the displacement and t is the time.

In subsequent chapters, formulas will be developed for finding quickly and easily derivatives of algebraic, circular, and exponential functions. In every case, however, the development of such formulas involves, directly or indirectly, the increment process employed in the preceding pages. This process, on account of its importance, is given below in general form.

Let

$$y = f(x).$$

Then

$$\begin{aligned} y + \Delta y &= f(x + \Delta x) \\ \Delta y &= f(x + \Delta x) - f(x) \\ \frac{\Delta y}{\Delta x} &= \frac{f(x + \Delta x) - f(x)}{\Delta x} \end{aligned}$$

[1] Other symbols for the derivative of y with respect to x are $D_x y$ and y'.

and

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{1}$$

19. Maxima and Minima. The algebraic sign of $\frac{dy}{dx}$, the derivative of y with respect to x, enables us to tell at once where the function y is increasing and where it is decreasing as x increases. For the derivative represents the slope of the tangent. If the slope is positive at a point, the function is increasing with x at that point. Similarly, if the slope is negative, the function is decreasing as x increases. Thus the function $y = x^2$ is a decreasing function when $x < 0$ and an increasing function when $x > 0$,

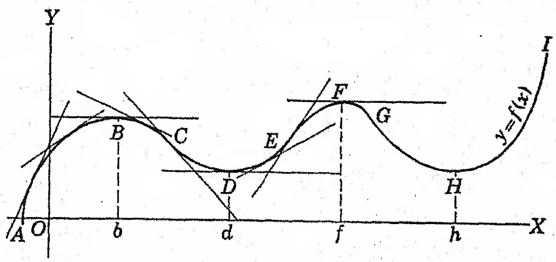

Fig. 14.

since the slope is equal to $2x$. When $x = 0$ the slope is zero and the tangent is parallel to the X-axis. Since the function is decreasing to the left of $x = 0$ and increasing to the right of this line, it follows that the function decreases to the value zero when $x = 0$ and then increases. This value zero is a minimum value of the function $y = x^2$. In general, we define minimum and maximum values of a function as follows:

Definition. Let $y = f(x)$, where $f(x)$ is any function of a single argument. *If y decreases to a value m as x increases and then begins to increase, m is called a minimum value of the function. If y increases to the value M as x increases and then begins to decrease, M is called a maximum value of the function.*

Thus in Fig. 14 if $ABDFHI$ is the graph of $y = f(x)$, the function increases to the value represented by the ordinate bB and then begins to decrease. bB is then a maximum value of the

function. Similarly, fF is another maximum value. dD and hH are minimum values of the function.

In referring to the graph of a function, points corresponding to maximum and points corresponding to minimum values of the function will be called, respectively, maximum and minimum points of the curve. B and F, Fig. 14, are maximum points and D and H are minimum points of the curve.

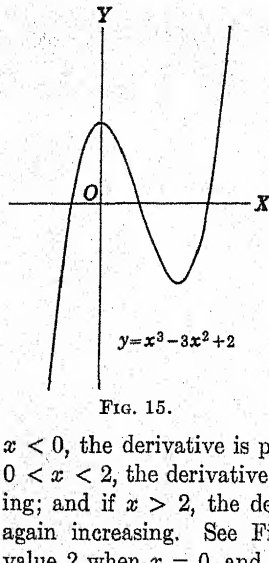

FIG. 15.

Thus, zero is a minimum value of $y = x^2$ or $(0, 0)$ is a minimum point on the curve $y = x^2$.

It will be noticed that a maximum value, as here defined, is not necessarily the largest value of the function, nor is a minimum value necessarily the smallest value of the function. A maximum value may even be less than a minimum value.

Illustration. The student will show that the derivative of $x^3 - 3x^2 + 2$ with respect to x is $3x(x - 2)$. If $x < 0$, the derivative is positive and the function is increasing; if $0 < x < 2$, the derivative is negative and the function is decreasing; and if $x > 2$, the derivative is positive and the function is again increasing. See Fig. 15. The function has a maximum value 2 when $x = 0$, and a minimum value -2 when $x = 2$.

Exercises

In each of the following exercises, find for what values of x the function is increasing; decreasing. Find the maximum and minimum values of the function, if there are any. Sketch a curve representing the function.

1. $y = -3x^3$.

2. $y = 4x^4$.

3. $y = -3x^4$.

4. $y = x^2 - 2x + 3$.

5. $y = 2x - x^2 + 1$.

6. $y = 2x^3 + 3x^2 - 36x$.

20. Illustration. As an application of the derivative consider the following problem.

The edge of a cube is increasing at the rate of 0.1 inch per minute. At what rate is the volume of the cube increasing when the edge is 7 inches long?

Let x be the length of the edge of the cube and y its volume. $\frac{dx}{dt}$, the rate of change of x, is given as 0.1 inch per minute and $\frac{dy}{dt}$, the rate of change of y, is to be found. Let t take on the increment Δt. Then x will take on an increment Δx and consequently y will take on an increment Δy. Now

$$y = x^3$$

and

$$\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

Dividing by Δt,

$$\frac{\Delta y}{\Delta t} = 3x^2\frac{\Delta x}{\Delta t} + 3x\frac{\Delta x}{\Delta t}\Delta x + \frac{\Delta x}{\Delta t}(\Delta x)^2.$$

Take the limit of each member of this equation as Δt approaches zero and note that Δx approaches zero with Δt. It follows that

$$\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$

Substituting $x = 7$ and $\frac{dx}{dt} = 0.1$,

$$\frac{dy}{dt} = 14.7.$$

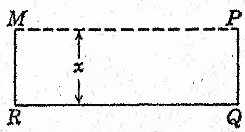

FIG. 16.

The volume is increasing at the rate of 14.7 cubic inches per minute at the instant when the edge is 7 inches in length.

21. Illustration. The solution of the following problem will further illustrate the use of the derivative.

Find the dimensions of the gutter with the greatest rectangular cross section which can be made from strips of tin 30 inches wide by bending up the edges to form the sides (see Fig. 16).

If the depth MR is determined, the width is also determined, since the sum of the three sides MR, PQ, and RQ is 30 inches.

If x represents the depth and y the area of the cross section of the gutter, we have

$$y = x(30 - 2x). \tag{1}$$

Equation (1) can be put in the form

$$y - 112.5 = -2(x - 7.5)^2,$$

which shows that the graph of (1) is a parabola (Fig. 17), with vertex at the point (7.5, 112.5).

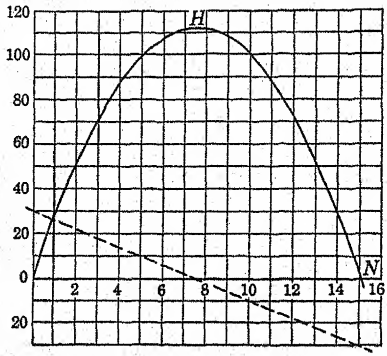

FIG. 17.

We find from equation (1), by the method of increments, that

$$\frac{dy}{dx} = -4(x - 7.5). \tag{2}$$

If $x < 7.5$, $\frac{dy}{dx}$ is positive and y is an increasing function. If $x > 7.5$, $\frac{dy}{dx}$ is negative and y is a decreasing function. When $x = 7.5$, $\frac{dy}{dx} = 0$. In this function the derivative changes from positive to negative values by passing through zero. Hence the function y, the area of the cross section, increases up to a certain value at $x = 7.5$ and then begins to decrease. The gutter will

have the greatest cross section when the depth is made 7.5 inches. The area of this greatest cross section is found to be 112.5 square inches by substituting 7.5 for x in equation (1). The graph of the derivative is shown by the dotted line in Fig. 17.

Exercises

In each of the following exercises from 1 to 6 inclusive, find: (*a*) the derivative of y with respect to x; (*b*) the values of x for which the function is increasing, decreasing; (*c*) the maximum and minimum points on the curve, if there are any.

1. $y = x^2 - 2x + 3.$

2. $y = 2x^3 - 3x^2 + 6.$

3. $y = 2x^2 + 3x + 6.$

4. $y = x^4 - 2x^2.$

5. $y = 3x^4 - 8x^3 - 6x^2 + 24x.$

6. $y = 3x^4 + 8x^3 - 6x^2 - 24x.$

7. Find the derivative of $\sqrt{x}$.

Solution. Let $y = \sqrt{x}$.

Then

$$y + \Delta y = \sqrt{x + \Delta x}$$

and

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}.$$

Rationalize the numerator:

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}.$$

As Δx approaches zero the right-hand side of this equation approaches $\frac{1}{\sqrt{x} + \sqrt{x}}$. Then

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{2\sqrt{x}},$$

or

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$

Find the derivative of y with respect to x in the following six exercises:

8. $y = \sqrt{x + 2}.$

9. $y = \sqrt{2x - 3}.$

10. $y = \sqrt{3x + 4}.$

11. $y = \frac{1}{\sqrt{x}}.$

12. $y = x^{\frac{3}{2}}.$

13. $y = \frac{1}{\sqrt{x + 2}}.$

14. The edge of a square is increasing uniformly at the rate of 0.2 inch per minute. At what rate is the area of the square increasing when the edge is 12 inches long?

15. Find the dimensions of the gutter with the greatest rectangular cross section which can be made from strips of tin 40 inches wide by bending up the edges to form the sides.

16. The radius of a circular plate is increasing uniformly at the rate of 0.03 inch per minute. At what rate is the area of the plate increasing when the radius is 10 inches long?

17. Find the dimensions of the largest rectangular field that can be enclosed by 160 rods of additional fencing if the field is to be adjacent to a fence already constructed along a straight railroad track.

18. The sides of an equilateral triangle are increasing at the uniform rate of 0.15 foot per minute. At what rate is the area of the triangle increasing when the sides are 2 feet long? When the altitude is 3 feet long?

19. A gutter is formed from a strip of tin 40 inches wide by bending up equal portions along each side. Each portion bent up makes an angle of 45° with the plane of the base extended on either side. Find the width of the parts turned up so that the area of the cross section of the gutter shall be a maximum.

20. The same as Exercise 19 except that the angles at the base are 60° instead of 45°.

CHAPTER II

LIMITS

In §18 the derivative was defined as the limit of a certain ratio. The word "limit" was used without giving its precise definition, as the reader was supposed to have a fair conception of the meaning of this term from previous courses in mathematics. However, since the entire subject of the calculus is based on limit processes, it is well to review the precise definition and to state certain theorems from the theory of limits.

22. Definition. *If a variable changes by an unlimited number of steps in such a way that, after a sufficiently large number of steps, the numerical value of the difference between the variable and a constant becomes and remains, for all subsequent steps, less than any preassigned positive constant, however small, the variable is said to approach the constant as a limit, and the constant is called the limit of the variable.*

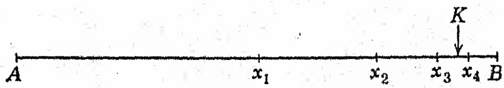

FIG. 18.

Illustration 1. Let AB, Fig. 18, be a line two units in length, and let x be the distance from A to a point on this line. Suppose that x increases from 0 by steps such that any value of x is greater than the preceding value by one-half of the difference between 2 and this preceding value, *i.e.*, by $\dfrac{2-x}{2}$. $x_1, x_2, x_3, x_4, \ldots$ are the end points of the portions of the line representing the successive values of x. Then the lengths $x_1B = 1$, $x_2B = \frac{1}{2}$, $x_3B = (\frac{1}{2})^2$, $x_4B = (\frac{1}{2})^3, \ldots, x_nB = (\frac{1}{2})^{n-1}$ are the successive differences between the constant length 2 and the variable length x. This

difference becomes and remains less than any preassigned length KB after a sufficient number of steps have been taken. This is true however small the length KB is chosen. Therefore, by the definition of the limit of a variable, 2 is the limit of the variable x.

Illustration 2. Consider the variable $y = x^2 - 2$ as x approaches the value 3 by any conveniently chosen unlimited number of steps. For example, give x the values

$$2.9,\ 2.99,\ 2.999,\ 2.9999,\ \ldots .$$

The corresponding values of $y = x^2 - 2$ are

$$6.41,\ 6.9401,\ 6.9940,\ 6.9994,\ \ldots .$$

When $x = 3$, $y = 7$. It is clear from the preceding set of values that the numerical value of the difference between 7 and $y = x^2 - 2$ becomes and remains less than any preassigned positive constant however small after a sufficiently large number of the steps by which x approaches 3 have been taken. Also if x is made to approach 3 by the steps

$$3.1,\ 3.01,\ 3.001,\ 3.0001,\ \ldots ,$$

the numerical value of the difference between 7 and $y = x^2 - 2$ becomes and remains less than any preassigned positive constant however small after a sufficiently large number of steps have been taken.

Therefore 7 is the limit of the variable $y = x^2 - 2$ as x approaches 3. The numerical value of the difference between the variable $y = x^2 - 2$ and the constant 7 can be made less than any preassigned positive constant however small by choosing x sufficiently near 3.

Illustration 3. Consider $y = \dfrac{1}{x-2}$. When x is given values nearer and nearer 2, for example, the values

$$1.9,\ 1.99,\ 1.999,\ \ldots ,$$

the corresponding values of $y = \dfrac{1}{x-2}$,

$$-10,\ -100,\ -1000,\ \ldots$$

become numerically larger and larger. If x approaches 2 by the steps

$$2.1,\ 2.01,\ 2.001,\ \ldots,\ \ldots,\ \ldots,$$

the corresponding values of $y = \dfrac{1}{x-2}$ are

$$10,\ 100,\ 1000,\ \ldots,\ \ldots,\ \ldots.$$

Indeed, the numerical value of $y = \dfrac{1}{x-2}$ can be made greater than any preassigned positive number however large by choosing x sufficiently near 2. The variable $y = \dfrac{1}{x-2}$ does not approach a limit as x approaches 2. Instead of doing so, it increases without limit.

If a variable changes by an unlimited number of steps in such a way that after a sufficiently large number of steps its numerical value becomes and remains, for all subsequent steps, greater than any preassigned positive number however large, the variable is said to become infinite. Illustration 3 of this section is an example of a variable which becomes infinite.

23. Notation. If in any limit process, the variable, say y, is a function of another variable, say x, the successive steps by which y changes are determined by those by which x changes. *If y approaches a limit A, as x approaches a limit a, we say that the limit of y as x approaches a is A, and write*

$$\lim_{x \to a} y = A.$$

The difference between the variable y and its limit, the constant A, can be made as small as we please in numerical value by choosing x sufficiently near a.

Thus (see *Illustration* 2, §22)

$$\lim_{x \to 3} (x^2 - 2) = 7$$

We also write (see *Illustration* 3, §22)

$$\lim_{x \to 2} \frac{1}{x-2} = \infty.$$

In this case a limit does not really exist. The form of expression is only a convenient way of saying that if x is taken sufficiently near 2 the value of y can be made to become and remain greater in numerical value than any preassigned positive number however large.

Further we write

$$\lim_{x \to \infty} \frac{1}{2 + \frac{1}{x}} = \frac{1}{2}.$$

This means that by taking x sufficiently large in numerical value, we can make the numerical value of the difference between $\frac{1}{2}$ and $\frac{1}{2 + \frac{1}{x}}$ less than any preassigned positive constant however small.

24. Infinitesimal. In the particular case where the limit of a variable is zero, the variable is said to be an *infinitesimal*. *An infinitesimal is a variable whose limit is zero.* Thus Δy and Δx which were used in §§**13**, **14**, and **15** are thought of as approaching zero and are infinitesimals. Hence the derivative (§**18**) is defined as the limit of the quotient of two infinitesimals. Infinitesimals are of fundamental importance in the calculus. Indeed, the subject is often called the infinitesimal calculus.

25. Theorems on Limits. The statements of four theorems concerning limits follow:

Theorem I. *The limit of the sum of two variables, each of which approaches a limit, is equal to the sum of their limits.*

Theorem II. *The limit of the difference of two variables, each of which approaches a limit, is equal to the difference of their limits.*

Theorem III. *The limit of the product of two variables, each of which approaches a limit, is equal to the product of their limits.*

Theorem IV. *The limit of the quotient of two variables, each of which approaches a limit, is equal to the quotient of their limits, provided the limit of the divisor is not zero.*

In Theorem IV, if the limit of the divisor is zero, the quotient of the limits has no meaning, since the operation of division by zero

is not defined. The quotient Q of two numbers A and B is defined as the number such that when it is multiplied by the divisor B, the product is the dividend A. Now if B is zero while A is not zero, there is no number which satisfies this requirement.

Proofs of Theorems I and IV will be given. Proofs of the other two theorems can be readily constructed by similar methods.

Proof of Theorem I: Let U and V be two variables whose limits are L and M, respectively. Denote the difference between the variable U and its limit L by u, and write

$$U = L + u. \tag{1}$$

Since U approaches L as a limit, it follows that the limit of u is zero. In like manner, if the difference between V and its limit M is v,

$$V = M + v, \tag{2}$$

where the limit of v is zero.

It is to be observed that u and v may be positive or negative. Adding (1) and (2),

$$U + V = L + M + u + v.$$

Hence, since $\lim u = 0$ and $\lim v = 0$,

$$\lim (U + V) = L + M.$$

Proof of Theorem IV: Using the same notation as in the proof of Theorem I, we write

$$\begin{aligned} \frac{U}{V} &= \frac{L + u}{M + v} \\ &= \frac{L}{M} + \frac{L + u}{M + v} - \frac{L}{M} \\ &= \frac{L}{M} + \frac{Mu - Lv}{M^2 + Mv}. \end{aligned}$$

Then

$$\lim \frac{U}{V} = \frac{L}{M} + \lim \frac{Mu - Lv}{M^2 + Mv}.$$

Let us consider the quotient $\frac{Mu - Lv}{M^2 + Mv}$. The limit of the numerator is zero. The denominator will ultimately become and remain greater than $\frac{M^2}{2}$. Hence the numerical value of $\frac{Mu - Lv}{M^2 + Mv}$ will ultimately be not greater than the numerical value of $\frac{2(Mu - Lv)}{M^2}$. But the limit of this quotient is zero. Hence $\lim \frac{Mu - Lv}{M^2 + Mv} = 0$. Consequently,

$$\lim \frac{U}{V} = \frac{L}{M}$$

if M is not zero.

26. The Indeterminate Form $\frac{0}{0}$. If in the quotient considered immediately preceding the proofs of Theorems I and IV, A is zero as well as B, any number will satisfy the requirements and Q is not determined. One encounters exactly this difficulty in seeking the value of $\frac{\sin x}{x}$ at $x = 0$. Its value is not determined at $x = 0$, but it is determined for all values of x different from zero. We define the value of $\frac{\sin x}{x}$ at $x = 0$ as the limit which it approaches as x approaches zero. In §55, it is shown that this limit is 1 if x is measured in radians. The student should construct on a large scale a graph of this function, giving to x the following values in radians: ± 1.5, ± 1.0, ± 0.5, ± 0.4, ± 0.3, ± 0.2, ± 0.1, ± 0.05.

The expression $\frac{\sin x}{x}$ is said to be *indeterminate* at $x = 0$ since any one of an infinite number of values can be assigned to it. The determination of its limiting value as x approaches zero is called the *evaluation of the indeterminate form.* The evaluation of the following indeterminate form is very simple:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4.$$

Exercises

1. Determine the following limits, if they exist:

(a) $\lim\limits_{x \to \frac{\pi}{2}} \cos x.$ (b) $\lim\limits_{x \to 0} \cot x.$

(c) $\lim\limits_{x \to 0} \sin \frac{\pi}{x}.$ Draw the curve for values of x between $-\pi$ and $+\pi$.

(d) $\lim\limits_{x \to 0} x \sin \frac{\pi}{x}.$

2. Evaluate the following indeterminate forms:

(a) $\left.\dfrac{x^2 - 9}{x - 3}\right|_{x \to 3}.$ (b) $\left.\dfrac{x^4 + 6x^2}{3x^3 + x^2}\right|_{x \to 0}.$

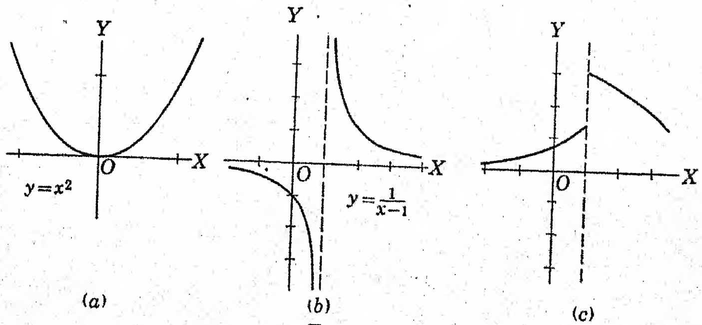

FIG. 19.

3. Find $\lim\limits_{x \to \infty} \dfrac{3x^2}{x}$; $\lim\limits_{x \to \infty} \dfrac{4x}{5x^2}$; $\lim\limits_{x \to \infty} \dfrac{4x^2}{5x^2}$; $\lim\limits_{x \to \infty} \dfrac{4x^2 + 3}{5x^2}$. Discuss the symbol $\dfrac{\infty}{\infty}$. Show that it is an indeterminate form.

27. Continuous and Discontinuous Functions. The curve of Fig. 19*a* is continuous everywhere while the curves of Fig. 19*b*, *c* are discontinuous at $x = 1$. A function $f(x)$ is said to be continuous at $x = a$ if $\lim\limits_{x \to a} f(x) = f(a)$. This condition is satisfied at all points in the case of the function of Fig. 19*a*, but it is not satisfied at $x = 1$ in the case of the functions of Fig. 19*b*, *c*.

Exercises

For what values of x, if any, are the following functions discontinuous? Sketch a graph in each case.

1. $y = \frac{1}{x}$.

2. $y = \frac{x}{3} + \frac{2}{x}$ (see §22).

3. $y = \sin x$.

4. $y = \sin \frac{\pi}{x}$.

5. $y = \tan x$.

6. $y = 2^x$.

7. $y = 2^{\frac{1}{x}}$

8. $y = \log x$.

9. $y = \frac{1}{(x-1)^2}$ (see §22).

10. $y = \frac{1}{x^2 - 5x + 6}$.

CHAPTER III

THE POWER FUNCTION

28. In Chapter I the derivative of a function was found by what may be called the fundamental method, *viz.*, by giving to the independent variable an increment, calculating the corresponding increment of the dependent variable, and finding the limit of the ratio of these increments as the increment of the independent variable approaches zero. This method is laborious and, since it will be necessary to find derivatives in a large number of problems, rules will be established by means of which the derivatives of certain functions can be written down at once. *The process of finding the derivative of a function is called differentiation.*

In this chapter we shall find the derivative of the power function, and study the function by means of this derivative.

The graphs of $y = x^n$, for various values of n, appear in Figs. 1, 2, 3, and 4. If n is positive, the curves go through the points (0, 0) and (1, 1), and are said to be of the parabolic type. In this case x^n is an increasing function of x in the first quadrant. If n is negative, the curves go through the point (1, 1) but do not go through the point (0, 0). They are asymptotic to both axes of coordinates. These curves are said to be of the hyperbolic type In this case x^n is a decreasing function of x in the first quadrant.

The law of the power function, as stated in §3, should be reviewed at this point.

29. Derivative of ax^n. Let $y = ax^n$, (1)

where n is at first assumed to be a positive integer.

Let x take on an increment Δx. Then y takes on an increment Δy and

$$\begin{aligned} y + \Delta y &= a(x + \Delta x)^n \\ &= a\left[x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}(\Delta x)^2 + \cdots + (\Delta x)^n\right] \end{aligned}$$

Subtract (1),

$$\Delta y = a\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots + (\Delta x)^{n-1}\right]\Delta x. \quad (2)$$

Divide by Δx,

$$\frac{\Delta y}{\Delta x} = a\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots + (\Delta x)^{n-1}\right].$$

Then

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = anx^{n-1},$$

or

$$\frac{dy}{dx} = anx^{n-1}. \quad (3)$$

That is,

$$\frac{\mathbf{d(ax^n)}}{\mathbf{dx}} = \mathbf{anx^{n-1}}. \quad (4)$$

If, in the expression $y = ax^n$, x is a function of another variable t, the derivative with respect to t is readily found. When t takes on an increment Δt, x takes on an increment Δx, and y takes on an increment Δy whose expression in terms of x and Δx is given in (2) above. Divide this expression by Δt and obtain

$$\frac{\Delta y}{\Delta t} = a\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots + (\Delta x)^{n-1}\right]\frac{\Delta x}{\Delta t}.$$

Take the limit of each member as Δt approaches zero and note that Δx approches zero with Δt.
Then

$$\frac{dy}{dt} = anx^{n-1}\frac{dx}{dt}, \quad (5)$$

or

$$\frac{\mathbf{d(ax^n)}}{\mathbf{dt}} = \mathbf{anx^{n-1}}\frac{\mathbf{dx}}{\mathbf{dt}}. \quad (6)$$

If t denotes time, equation (5) expresses the rate of change of

y in terms of x and the rate of change of x. Equation (3) expresses the rate of change of y with respect to x.

Equation (3) can be written in the form

$$\frac{d(ax^n)}{dx} = \frac{anx^n}{x} = n\frac{y}{x}. \tag{7}$$

The geometrical meaning of formula (7) is shown by Fig. 20. The fraction $\frac{y}{x}$ is the slope of the radius vector OP from the origin to the point P on the curve. Formula (7) states that the slope, at any point of the graph of the function, $y = ax^n$, is n times the slope of the radius vector OP. Thus, if $n = 1$, $y = ax^n$ reduces to a straight line through the origin, and the line has the same slope as OP. If $n = 2$ the curve is the parabola $y = ax^2$, and the slope of the curve is always twice that of OP. If $n = -1$, the curve is the rectangular hyperbola, $y = \frac{a}{x}$, and the slope of the curve is the negative of the slope of OP.

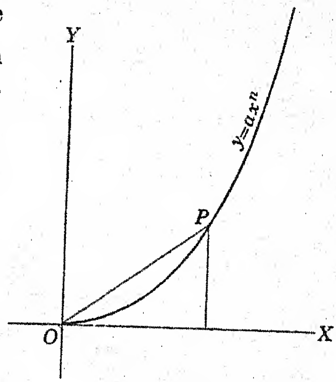

FIG. 20.

In like manner, (5) can be written

$$\frac{dy}{dt} = n\frac{y}{x}\frac{dx}{dt},$$

or

$$\frac{1}{y}\frac{dy}{dt} = n\frac{1}{x}\frac{dx}{dt}. \tag{8}$$

$\frac{1}{y}\frac{dy}{dt}$ is the rate of change of y per unit y. Equation (8) states that, in the case of the power function $y = ax^n$, the rate of change of y per unit y is equal to n times the rate of change of x per unit x (see *Illustration* 3, §30).

The formulas (4) and (6) were derived on the assumption that n is a positive integer. They hold for all values of the constant n. This will be proved later (see §**91**). For the present we shall assume these formulas to be true for all values of the constant exponent n.

Illustrations.

1. $y = 7x^2$. $\qquad \dfrac{dy}{dx} = 14x.$

2. $y = 5x^{\frac{3}{2}}$. $\qquad \dfrac{dy}{dx} = \frac{15}{2}x^{\frac{1}{2}}.$

3. $y = 4x^3$. $\qquad \dfrac{dy}{dt} = 12x^2\dfrac{dx}{dt}.$

4. $y = \dfrac{3}{x^2} = 3x^{-2}$. $\qquad \dfrac{dy}{dx} = -6x^{-3} = -\dfrac{6}{x^3}.$

5. $s = 5t^3$. $\qquad \dfrac{ds}{dt} = 15t^2.$

6. $y = 3x^2$. $\qquad \dfrac{dy}{dx} = 2\dfrac{y}{x}.$

$$\frac{1}{y}\frac{dy}{dt} = 2\frac{1}{x}\frac{dx}{dt}.$$

Exercises

Find $\dfrac{dy}{dx}$:

1. $y = x^2$.

2. $y = 5x^3$.

3. $y = 4x^{\frac{1}{3}}$.

4. $y = 3x^5$.

5. $y = -\frac{3}{5}x^4$.

6. $y = \dfrac{5}{x^3}$.

7. $y = 2x^{1.2}$.

8. $y = \dfrac{3}{x^2}$.

9. $y = 10\sqrt{x}$.

10. $y = -3x^5$.

11. $y = \dfrac{5}{\sqrt{x}}$.

12. $y = 4x^{2.3}$.

13. $y = -2x^3$.

14. $y = \frac{1}{5}x^{10}$.

15. $y = -4x^{\frac{3}{2}}$.

Find $\dfrac{dy}{dt}$, x being considered a function of t:

16. $y = 2x^3$.

17. $y = -2\sqrt{x}$.

18. $y = \pi x^2$.

19. $y = \dfrac{3}{x}$.

20. $y = 3x^5$.

21. $y = -\dfrac{3}{\sqrt{x}}$.

22. $y = \dfrac{4}{x^2}$.

23. $y = \frac{4}{3}\pi x^3$.

Find the derivative of each of the following functions with respect to the independent variable in terms of which it is expressed:

24. $s = 5t^4$.

25. $z = 3w^5$.

26. $u = 3x^4$.

27. $s = \dfrac{3}{\sqrt{t}}$.

28. $y = 2t^3$.

29. $u = 6t^2$.

30. $z = 3y^2$.

31. $w = 5\sqrt{z}$.

Apply formulas (7) and (8) in the four following exercises:

32. $y = 5x^2$.

33. $y = 4x^3$.

34. $y = 2\sqrt{x}$.

35. $y = 5\sqrt{x^3}$.

36. Sketch the curve $xy = a$, where a is a positive constant. Let P be any point on the curve and let APB be the tangent to the curve at the point P cutting the X-axis at A and the Y-axis at B. Show that a circle with center at P and radius equal to OP passes through the points A and B.

37. Prove that the area of the triangle AOB of Exercise 36 is independent of the position of the point P.

38. Find the equation of the line tangent to the curve $y = x^3$ at the point whose abscissa is -2.

30. Rates of Change. If $y = ax^n$, the rate of change of y is expressed in terms of x and the rate of change of x, by equation (5), §29, *viz.*,

$$\frac{dy}{dt} = anx^{n-1}\frac{dx}{dt}.$$

Illustration 1. The side of a square is increasing at the uniform rate of 0.2 inch per second. Find the rate at which the area is increasing when the side is 10 inches long.

Let x be the length of the side, and y the area of the square. Then $\dfrac{dx}{dt} = 0.2$ and $\dfrac{dy}{dt}$ is the rate of increase of the area. To find

this rate of increase, differentiate the function $y = x^2$ with respect to t.

$$\frac{dy}{dt} = 2x\frac{dx}{dt}.$$

Since

$$\frac{dx}{dt} = 0.2,$$

$$\frac{dy}{dt} = 0.4x.$$

When $x = 10$, $\frac{dy}{dt} = 4$. The area is increasing at the rate of 4 square inches per second. When $x = 13$, $\frac{dy}{dt} = 5.2$, the rate of change of the area at this instant.

Illustration 2. A spherical soap bubble is being inflated at the rate of 0.2 cubic inch per second. Find the rate at which the radius is increasing when it is 1.5 inches long.

Let r be the radius, and V the volume of the bubble. $\frac{dV}{dt} = 0.2$ and $\frac{dr}{dt}$, the rate of increase of the radius, is to be found.

$$V = \tfrac{4}{3}\pi r^3,$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}.$$

From which

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}.$$

Since

$$\frac{dV}{dt} = 0.2, \text{ and } r = 1.5,$$

$$\frac{dr}{dt} = \frac{1}{20\pi(1.5)^2} = 0.0071 \text{ inch per second.}$$

Illustration 3. The edges of a cube are increasing at a certain rate. Compare the rate of increase of the volume per unit volume with the rate of increase of the edges per unit length.

Let x be the length of an edge and y the volume.

Then

$$y = x^3$$

and

$$\frac{1}{y}\frac{dy}{dt} = 3\,\frac{1}{x}\frac{dx}{dt}.$$

Thus the rate of increase of the volume per unit volume is three times the rate of increase of the edge per unit length.

If at a certain instant the edges are 10 inches long and increasing at the rate of 0.1 inch per minute,

$$\frac{1}{x}\frac{dx}{dt} = 0.01 \text{ inch per inch per minute}$$

and

$$\frac{1}{y}\frac{dy}{dt} = 0.03 \text{ cubic inch per cubic inch per minute.}$$

Illustration 4. A gas in a cylinder is expanding in accordance with Boyle's law, $pv = K$. Find the rate at which the pressure is changing when the volume is 1000 cubic inches and the pressure is 60 pounds per square inch, if at this instant the volume is increasing at the rate of 2 cubic inches per second.

By differentiating

$$p = \frac{K}{v} \tag{1}$$

with respect to t, we obtain

$$\frac{dp}{dt} = \frac{-K}{v^2}\frac{dv}{dt}. \tag{2}$$

Eliminate K between (1) and (2):

$$\frac{dp}{dt} = -\frac{p}{v}\frac{dv}{dt}. \tag{3}$$

On substituting in (3) the values of p, v, and $\frac{dv}{dt}$ as given in the statement of the problem, we obtain

$$\frac{dp}{dt} = -0.12.$$

The minus sign indicates that the pressure is decreasing as t increases. Thus, under the conditions of the problem, the pressure is decreasing at the rate of 0.12 pound per square inch per second.

Exercises

1. The radius of a spherical soap bubble is increasing at the rate of 0.2 inch per second. At what rate is the surface of the bubble increasing when the radius is 2 inches long? At what rate is the volume increasing at the same instant?

Find the rate of increase of the surface per unit surface and the rate of increase of the volume per unit volume.

2. A spherical balloon is being inflated at the rate of 5 cubic inches per second. At what rate is the surface of the balloon increasing when the radius is 3 inches?

3. Water is flowing into a V-shaped trough 4 feet long at the rate of 400 cubic inches per minute. The angle between the sides of the trough is 60°. Find the rate at which the depth of the water is increasing when it is 6 inches deep.

4. At a certain instant the sides of a square are increasing at the rate of 0.2 inch per minute and the area at the rate of 20 square inches a minute. What is the length of a side of the square at this instant?

5. Water is flowing into an inverted right circular cone whose vertical angle is 60°. The depth of the water is increasing at the rate of 1 foot per minute when it is 4 feet deep. At what rate is water flowing into the cone?

6. An opaque circular disk 4 inches in diameter is held between a screen and a light which may be imagined concentrated at a point. The distance between the screen and the light is 20 feet. The disk is kept parallel to the screen and moved toward the light at the rate of 2 feet per minute. At what rate is the area of the shadow of the disk increasing when the disk is 6 feet from the light?

7. Liquid is being drawn through a straw from a conical cup at the rate of 2 cubic inches per second, the angle of the cone being 90°. At what rate is the surface of the liquid being lowered when the depth of the liquid in the cup is 4 inches?

8. A gas is expanding in accordance with Boyle's law, $pv = C$. At a certain instant $p = 4000$ pounds per square foot and $v = 5$ cubic feet. If at this instant the volume is increasing at the rate of 1 cubic foot per minute, at what rate is the pressure changing?

9. The gas referred to in Exercise 8 is expanding in accordance with the adiabatic law, $pv^{1.4} = C$. At what rate is the pressure changing?

10. A gas is expanding in accordance with the law $pv^{1.4} = K$. At a certain instant $p = 3000$ pounds per square foot and $v = 5$ cubic feet. If at this instant the volume is increasing at the rate of 1 cubic foot per minute, at what rate is the pressure changing?

11. A gas is expanding in accordance with the law $pv^{1.48} = K$. At a certain instant the pressure is 60 pounds per square inch and the volume is 5 cubic feet. At this instant the pressure is decreasing at the rate of 0.2 pound per square inch per second. Find the rate at which the volume is changing.

12. A spherical toy balloon is being inflated. Find the radius of the balloon when the rate of increase of its volume, measured in cubic inches per second, is equal to the rate of increase of its surface, measured in square inches per second.

31. The Derivative of the Sum of a Function and a Constant. Sketch the graph of a function, for example, the graph of $y = x^2$. Then on the same set of axes sketch graphs of $y = x^2 + C$, giving to C several values such as -2, -1, 1, 2, 3, etc. Draw any vertical line AB.

Since each of the several curves is the curve $y = x^2$ translated in a direction parallel to the Y-axis, it is obvious that the slopes of their respective tangent lines at the points of intersection of the curves with the line AB are equal. In other words, the derivative of $x^2 + C$, for any given value of x, is independent of the value assigned to C, and is equal to the derivative of the function when C is zero.

The same reasoning will hold for any function, $f(x)$. Thus we have

$$\frac{d[f(x) + C]}{dx} = \frac{d[f(x)]}{dx}. \tag{1}$$

In particular,

$$\frac{d[ax^n + C]}{dx} = \frac{d[ax^n]}{dx} = anx^{n-1}.$$

An analytic derivation of (1) will now be given. Let

$$y = f(x) + C.$$

Then

$$\begin{aligned} y + \Delta y &= f(x + \Delta x) + C \\ \Delta y &= f(x + \Delta x) + C - f(x) - C \\ &= f(x + \Delta x) - f(x) \end{aligned}$$

and

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

or

$$\frac{d[f(x) + C]}{dx} = \frac{d[f(x)]}{dx}.$$

Illustration. If $y = 5x^3 + 7$,

$$\frac{dy}{dx} = \frac{d(5x^3)}{dx} = 15x^2.$$

Exercises

Differentiate:

1. $y = 3x^2 + 2.$
2. $y = 5\sqrt{x} + 4.$
3. $y = 2x^3 - 3.$
4. $y = 2t^4 + 7.$
5. $y = \dfrac{3}{x^2} + 5.$
6. $s = 16t^2 + 5.$
7. $s = 2\sqrt{t^3} + 6.$
8. $s = \dfrac{2}{x^3} - 4.$
9. $y = -4x^5 + 6.$
10. $y = -3x^4 + 2.$
11. $y = 7x^2 - 3.$
12. $y = 4x^3 + 5.$
13. $y = \dfrac{1}{x} + 2.$
14. $y = -\frac{1}{5}x^5 + 3.$
15. $y = \frac{1}{5}x^5 + 2.$
16. $y = \frac{3}{7}x^4 - 5.$

32. The Derivative of au^n. If $y = au^n$, where u is a function of x, it follows from (5), §29, that

$$\frac{dy}{dx} = anu^{n-1}\frac{du}{dx},$$

or

$$\frac{d(au^n)}{dx} = anu^{n-1}\frac{du}{dx}.$$

Illustrations.

1. $$\frac{d[5(x^2+3)^3]}{dx} = 5 \cdot 3(x^2+3)^2 \frac{d(x^2+3)}{dx} = 15(x^2+3)^2\, 2x$$
$$= 30x(x^2+3)^2.$$

2. $$\frac{d[2(x^2+4)^4+10]}{dx} = \frac{d[2(x^2+4)^4]}{dx} = 2 \cdot 4(x^2+4)^3 \frac{d(x^2+4)}{dx}$$
$$= 2 \cdot 4(x^2+4)^3\, 2x = 16x(x^2+4)^3.$$

3. If $y = (2x^2+1)^2$, find $\frac{dy}{dt}$.
$$\frac{dy}{dt} = \frac{d(2x^2+1)^2}{dt} = 2(2x^2+1)\frac{d(2x^2+1)}{dt}$$
$$= 2(2x^2+1) \cdot 2 \cdot 2x\frac{dx}{dt} = 8x(2x^2+1)\frac{dx}{dt}.$$

4. If $y = (x^2+1)^{\frac{3}{2}}$,
$$\frac{dy}{dx} = \tfrac{3}{2}(x^2+1)^{\frac{1}{2}}2x = 3x(x^2+1)^{\frac{1}{2}},$$

and
$$\frac{dy}{dt} = \tfrac{3}{2}(x^2+1)^{\frac{1}{2}}2x\frac{dx}{dt} = 3x(x^2+1)^{\frac{1}{2}}\frac{dx}{dt}.$$

Exercises

Find $\frac{dy}{dx}$:

1. $y = (3x^2+4)^4$.
2. $y = (3x^2-2)^5$.
3. $y = 4(2-3x^2)^3$.
4. $y = 3(5-2x^2)^4+2$.
5. $y = \sqrt{4-2x^2}$.
6. $y = \sqrt{5-3x^2}$.
7. $y = 3(x^2+2)^{\frac{1}{3}}$.
8. $y = 4(2x^3-5)^{\frac{1}{3}}$.
9. $y = \frac{5}{\sqrt{2-3x^2}}$.
10. $y = \frac{6}{\sqrt{3-2x^3}}$.
11. $y = (2-x^2)^{-2}$.
12. $y = (2x^3+5)^{-2}$.
13. $y = 5(x-7)^{-3}$.
14. $y = 2(3x+8)^{\frac{3}{2}}$.
15. $y = 5(2x-3)^{-\frac{2}{3}}+4$.
16. $y = 2+\sqrt{4-x^2}$.
17. $y = 5-\sqrt{3x^3-2}$.
18. $y = (x^{\frac{1}{2}}+2)^{\frac{1}{2}}+3$.
19. $y = \frac{3}{x-2}$.
20. $y = \frac{5}{(2-3x^2)^2}$.

33. The Derivative of a Constant. Let $y = c$, where c is a constant. Corresponding to any Δx, $\Delta y = 0$, and consequently

$$\frac{\Delta y}{\Delta x} = 0,$$

and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0,$$

or

$$\frac{dy}{dx} = 0.$$

The derivative of a constant is zero.

Interpret this result geometrically.

34. The Derivative of the Sum of Two Functions. Let

$$y = u + v,$$

where u and v are functions of x.

Let x take on an increment Δx. Then u, v, and y take on the increments Δu, Δv, and Δy, respectively.

$$\begin{aligned} y + \Delta y &= u + \Delta u + v + \Delta v \\ \Delta y &= \Delta u + \Delta v \\ \frac{\Delta y}{\Delta x} &= \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \\ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}. \end{aligned}$$

That is,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx},$$

or

$$\frac{d(\mathbf{u} + \mathbf{v})}{d\mathbf{x}} = \frac{d\mathbf{u}}{d\mathbf{x}} + \frac{d\mathbf{v}}{d\mathbf{x}}.$$

The derivative of the sum of two functions is equal to the sum of their derivatives.

The student will observe that the proof given can be extended to the sum of three, four, or any finite number of functions.

Illustrations.

1. $\dfrac{d(6x + 15x^2)}{dx} = \dfrac{d(6x)}{dx} + \dfrac{d(15x^2)}{dx} = 6 + 30x.$

2. $\dfrac{d(2\sqrt{x} + 3x^2 + 4)}{dx} = \dfrac{d(2\sqrt{x})}{dx} + \dfrac{d(3x^2)}{dx} + \dfrac{d(4)}{dx} = \dfrac{1}{\sqrt{x}} + 6x.$

3. $\dfrac{d(t^2 + 2t^3 + 3)}{dt} = 2t + 6t^2.$

Exercises

In Exercises 1 to 10 inclusive find $\dfrac{dy}{dx}$, and in Exercises 11 to 20 find $\dfrac{dy}{dt}$.

1. $y = 4x^3 - 3x^2 + 5x - 11.$

2. $y = 2x^3 - 4x^2 + 5.$

3. $y = \sqrt{3x^2 - 2x + 4}.$

4. $y = \sqrt{2x^2 - 3x - 5}.$

5. $y = 2\sqrt{x} - \dfrac{1}{x^2} + 3.$

6. $y = 3\sqrt{x} - \dfrac{1}{x} + \dfrac{1}{x^2} + 2.$

7. $y = \sqrt{x - 1} + \sqrt{2x + 3}.$

8. $y = \sqrt{2x - 1} - \sqrt{3x - 2}.$

9. $y = \dfrac{3}{x^2 - 2x + 3}.$

10. $y = \dfrac{5}{x^2 + 3x - 7}.$

11. $y = (2x^2 - 3x + 2)^{-2}.$

12. $y = 3(x^2 + 4x - 5)^{-1}.$

13. $y = (x^2 + 2x - 3)^{\frac{1}{2}}.$

14. $y = (x^2 - 2x + 3)^{\frac{1}{2}}.$

15. $y = \dfrac{5}{(2 - 4x + 6x^2)^2}.$

16. $y = \dfrac{6}{\sqrt{x^2 - 3x + 2}}.$

17. $y = 2x + \sqrt{9 - x^2}.$

18. $y = 3x - \sqrt{x^2 + 7x - 1}.$

19. $y = (ax^2 + bx + c)^2.$

20. $y = (ax^3 + bx^2 + cx + d)^3.$

Find for what values of x the functions of Exercises 21, 22, 23, and 24 are increasing; decreasing. Find the maximum and minimum values of the functions, if there are any.

21. $2x^3 - 3x^2 - 12x - 5.$

22. $3x^4 - 4x^3 + 1.$

23. $x^{\frac{2}{3}}.$

24. $(x - 1)^{\frac{2}{3}} + 1.$

25. Find the area of the largest rectangle that can be inscribed in an equilateral triangle of sides 10 inches in length, if the base of the rectangle rests upon one side of the triangle.

HINT. $A = \frac{\sqrt{3}}{2}(10x - x^2)$, where A is the area of the rectangle and x the length of the base.

26. Find the area of the largest rectangle that can be inscribed in an isosceles right triangle whose hypotenuse is 10 inches in length, if the base of the rectangle rests upon the hypotenuse of the triangle.

27. The length of a rectangle is increasing at the rate of 0.2 inch per minute, and is always equal to double the width. Find the rate at which the area of the rectangle is increasing when the diagonal is 10 inches.

35. Differentiation of Implicit Functions. The derivative of one variable with respect to another can be found from an equation connecting the variables without solving the equation for either variable, for, if the variables occurring in the equation are x and y, y is a function of x, even though its explicit form may not be known, and the usual rules for finding the derivative of functions can be applied to each member of the equation.

The following example will illustrate the process.

Illustration. Let $x^2 + y^2 = a^2$. Find $\frac{dy}{dx}$.

The left-hand member of the given equation is the sum of two functions of x, since y is a function of x. Further, the derivative of the left-hand member is equal to the derivative of the right-hand member. The derivative of the latter is in this case zero, since the right-hand member is constant. On differentiating the left-hand member as the sum of two functions, we obtain

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

When the derivative is found by differentiating each member of an equation in the implicit form, as in the foregoing illustration, the operation is called implicit differentiation.

Exercises

1. Draw the circle $x^2 + y^2 = a^2$ and show geometrically that the slope of the tangent at the point (x, y) is $-\frac{x}{y}$.

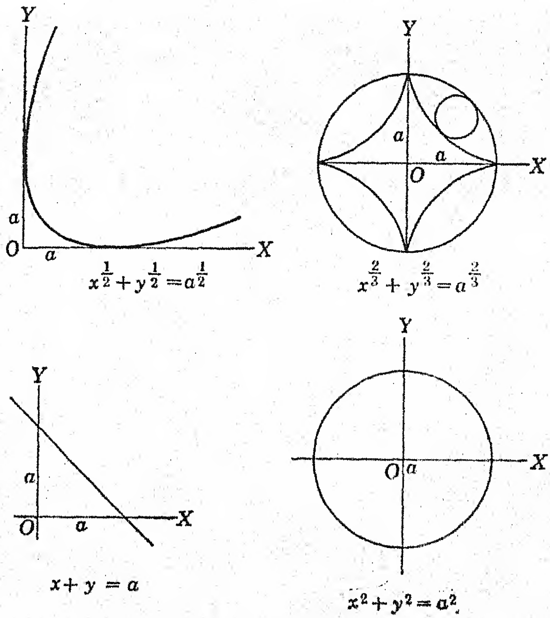

Fig. 21.

2. Solve the equation of Exercise 1 for y and find $\frac{dy}{dx}$.

From the following equations find $\frac{dy}{dx}$ by implicit differentiation:

3. $3x^2 + 4y^2 = 12$.

4. $x^2 - y^2 = a^2$.

5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (Do not clear of fractions.)

If y is an implicit function of x expressed by an equation of the form

$$x^n + y^n = a^n, \tag{1}$$

differentiation gives

$$nx^{n-1} + ny^{n-1}\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\left[\frac{x}{y}\right]^{n-1}. \tag{2}$$

The equation (1) includes a number of important special cases. The graphs corresponding to the following values of n are shown in Fig. 21. For

$n = \frac{1}{2}$, $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, a parabola,
$n = \frac{2}{3}$, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, an important hypocycloid,
$n = 1$, $x + y = a$, a straight line,
$n = 2$, $x^2 + y^2 = a^2$, a circle.

The graph of (1) passes through the points $(0, a)$ and $(a, 0)$ if n is positive.

6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

7. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

8. $x^3 + y^3 = a^3$.

9. $x^{\frac{3}{2}} + y^{\frac{3}{2}} = a^{\frac{3}{2}}$.

In Exercises 10, 11, 12, and 13 find $\frac{dy}{dt}$ in terms of x, y, and $\frac{dx}{dt}$:

10. $x^2 + y^2 = a^2$,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Hence,

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

11. $3x^2 + 4y^2 = 12$.

12. $x^2 - y^2 = 10$.

13. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

14. A point is moving on the arc of the circle $x^2 + y^2 = 100$. When the point is above the X-axis and 4 units to the right of the Y-axis, its abscissa is increasing at the rate of 2 units per second. At what rate is the ordinate changing?

15. A point is moving on the arc of the hyperbola $x^2 - y^2 = 75$. At a certain instant the point is in the first quadrant and is moving away from the X-axis twice as fast as it is moving away from the Y-axis. Find the position of the point at this instant.

16. Find the equations of the lines tangent to the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 8$ at the points where the curve is cut by the line $y = x$.

17. A point moves upward along the line $x = 5$ with a uniform speed of 3 units per second. At what rate is its distance from the origin changing when the point is 7 units below the X-axis?

18. A point moves to the right along the line $y = 5$ with a uniform speed of 4 units per second. At what rate is its distance from the point (3, 10) changing when it is 5 units to the left of the Y-axis? When it is 5 units to the right of the Y-axis?

19. The upper end of a ladder, 16 feet long, rests against a vertical wall and the lower end rests on a horizontal pavement. How fast is the upper end of the ladder moving downward when it is 10 feet above the pavement, if the lower end is drawn out at the rate of 2 feet per second?

20. A point is moving on the arc of the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2$. How many times faster is the point moving away from the Y-axis than from the X-axis when $x = 9$?

36. Antiderivative. Integration. Let it be required to find the equation of the curve whose slope at any point is twice the abscissa of that point and which passes through the point (2, 7).

The fact that the slope at any point is twice the abscissa of that point is expressed by the equation

$$\frac{dy}{dx} = 2x. \tag{1}$$

The equation of the curve will be of the form $y = f(x)$, where $f(x)$ is a function whose derivative with respect to x is $2x$. It is clear that x^2 is such a function, that $x^2 + 3$ is another, that $x^2 - 2$ is a third, and so on. Indeed, the derivative of any function of the form $x^2 + C$, where C is any constant, is equal to $2x$. Hence, any curve of the family of curves

$$y = x^2 + C \tag{2}$$

satisfies the requirement, that the slope at any point is equal to twice the abscissa of that point. The curves represented by equation (2) are parabolas, any one of which can be obtained by translating the parabola $y = x^2$ parallel to the Y-axis, upward if C is positive, downward if C is negative.

Since the curve is to pass through the point (2, 7), the coordinates of this point must satisfy the equation. Substitution in (2) gives

$$7 = 4 + C, \text{ or } C = 3.$$

Hence

$$y = x^2 + 3$$

is the equation of the curve through the point (2, 7) whose slope at any point is equal to twice the abscissa of that point.

The foregoing illustration introduces a new type of problem, *viz.*, that of finding a function whose derivative is given. A function whose derivative is equal to a given function is called an *antiderivative*, or *integral*, of the given function. From the illustration it is clear that any given function which has one antiderivative has an unlimited number of antiderivatives which differ from each other only by an additive constant. This latter fact is indicated in obtaining the antiderivative of a given function by writing down the variable part of the antiderivative and adding to it a constant C. In a given application this constant will be determined by supplementary conditions as in the illustration at the beginning of this section.

The process of finding the antiderivative of a given function is called *integration*. The arbitrary constant C is called *the constant of integration.*

Illustrations. The student will verify the results by differentiating them with respect to x or with respect to t and obtaining the expression given.

1. If $\dfrac{dy}{dx} = 3x^2,$ $\qquad y = x^3 + C.$

2. If $\dfrac{dy}{dx} = x^2,$ $\qquad y = \frac{1}{3}x^3 + C.$

3. If $\dfrac{dy}{dx} = 3x^2 + 2x,$ $\qquad y = x^3 + x^2 + C.$

4. If $\dfrac{dy}{dx} = 3x^2 + 2x + 7,$ $\qquad y = x^3 + x^2 + 7x + C.$

5. If $\dfrac{dy}{dt} = (x^2 + x + 7)\dfrac{dx}{dt},$ $\qquad y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + 7x + C.$

Exercises

Integrate the following ten functions:

1. $\dfrac{dy}{dx} = 5x^2.$

2. $\dfrac{dy}{dx} = 4x^3.$

3. $\dfrac{dy}{dt} = 4x^3 \dfrac{dx}{dt}.$

4. $\dfrac{dy}{dt} = 3x^2 \dfrac{dx}{dt},$

5. $\dfrac{dy}{dx} = 3x^2 + 2x - 6.$

6. $\dfrac{dy}{dt} = (3x^2 + 2x + 6) \dfrac{dx}{dt}.$

7. $\dfrac{dy}{dt} = (ax + b) \dfrac{dx}{dt}.$

8. $\dfrac{dy}{dx} = 3x^2 - 2x^{\frac{1}{2}} + 7.$

9. $\dfrac{dy}{dx} = 10x^{-2} + 2x^{-3} - x + 7.$

10. $\dfrac{dy}{dt} = (x^{\frac{1}{3}} + x^{-\frac{1}{3}}) \dfrac{dx}{dt}.$

Illustrations.

6. $\dfrac{dy}{dx} = 3(x^2 + 2)^2 2x.$

The right-hand side is in the form, $nu^{n-1} \dfrac{du}{dx}$, where n is 3, and u is $(x^2 + 2)$. Since the integral of $nu^{n-1} \dfrac{du}{dx}$ is $u^n + C$,

$$y = (x^2 + 2)^3 + C.$$

7. $\dfrac{dy}{dx} = (x^2 - 5)^3 \, 2x = \frac{1}{4}[4(x^2 - 5)^3 \, 2x].$

$$y = \tfrac{1}{4}(x^2 - 5)^4 + C.$$

8. $\dfrac{dy}{dx} = x(x^2 - 1)^5 = \frac{1}{6} \cdot \frac{1}{2}[6(x^2 - 1)^5 2x].$

$$y = \tfrac{1}{12}(x^2 - 1)^6 + C.$$

9. $\dfrac{dy}{dt} = x^2(3 - x^3)^5 \dfrac{dx}{dt} = -\frac{1}{6} \cdot \frac{1}{3}\left[6(3 - x^3)^5 \left(-3x^2 \dfrac{dx}{dt}\right)\right].$

$$y = -\tfrac{1}{18}(3 - x^3)^6 + C.$$

10. $\dfrac{dy}{dt} = (x^2 - 2x + 3)^{-3}(x - 1) \dfrac{dx}{dt}$

$$= -\tfrac{1}{2} \cdot \tfrac{1}{2}\left[-2(x^2 - 2x + 3)^{-3}(2x - 2) \frac{dx}{dt}\right].$$

$$y = \frac{-1}{4(x^2 - 2x + 3)^2} + C.$$

Exercises

Integrate:

11. $\dfrac{dy}{dx} = x\sqrt{x^2 - 1}$. *Ans.* $y = \frac{1}{3}(x^2 - 1)^{\frac{3}{2}} + C$.

12. $\dfrac{dy}{dx} = (2x^3 + 3x^2)^{\frac{1}{3}}(x^2 + x)$. *Ans.* $y = \frac{1}{8}(2x^3 + 3x^2)^{\frac{4}{3}} + C$.

13. $\dfrac{dy}{dx} = (x + 1)^{\frac{1}{3}}$. *Ans.* $y = \frac{3}{4}(x + 1)^{\frac{4}{3}} + C$.

14. $\dfrac{dy}{dx} = (2 - x^2)^2 x$.

15. $\dfrac{dy}{dx} = x\sqrt{4 - x^2}$.

16. $\dfrac{dy}{dt} = x\sqrt{x^2 - 3}\,\dfrac{dx}{dt}$.

17. $\dfrac{dy}{dx} = x(3 - x^2)^3$.

18. $\dfrac{dy}{dt} = x\sqrt{9 - x^2}\,\dfrac{dx}{dt}$.

19. $\dfrac{dy}{dt} = 2x(x^2 + 2)^2\,\dfrac{dx}{dt}$.

20. $\dfrac{dy}{dx} = 3x^2(x^3 + 2)^3$.

21. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 - 4}}$.

22. $\dfrac{dy}{dx} = \dfrac{x^2}{\sqrt{x^3 - 2}}$.

23. $\dfrac{dy}{dt} = x(5 - x^2)^4\,\dfrac{dx}{dt}$.

24. $\dfrac{dy}{dt} = x^2(2 - x^3)^4\,\dfrac{dx}{dt}$.

25. $\dfrac{dy}{dt} = \dfrac{x}{(x^2 + 4)^3}\,\dfrac{dx}{dt}$.

26. $\dfrac{dy}{dx} = \dfrac{x}{(1 - x^2)^2}$.

27. $\dfrac{dy}{dx} = x^2(4 - x^3)^2$.

28. $\dfrac{dy}{dx} = x^3(2 - x^4)^2$.

29. $\dfrac{dy}{dx} = (x^2 + 4x - 7)^3(x + 2)$.

30. $\dfrac{dy}{dx} = (x^2 - 6x + 1)^2(x - 3)$.

31. $\dfrac{dy}{dx} = (x + 2)\sqrt{x^2 + 4x + 2}$.

32. $\dfrac{dy}{dx} = \dfrac{x - 3}{\sqrt{x^2 - 6x + 1}}$.

33. $\dfrac{dy}{dx} = \dfrac{x + 2}{\sqrt{x^2 + 4x - 7}}$.

34. $\dfrac{dy}{dt} = \dfrac{x^2 - 1}{\sqrt{x^3 - 3x + 2}}\,\dfrac{dx}{dt}$.

35. $\dfrac{dy}{dx} = \dfrac{x + 1}{\sqrt{5 - x^2 - 2x}}$.

36. $\dfrac{dy}{dt} = (2 - 3x^2)^4 x\,\dfrac{dx}{dt}$.

37. $\dfrac{dy}{dx} = (3 - 2x^2)^4 x$.

38. $\dfrac{dy}{dx} = (4x - 1)^3$.

39. $\dfrac{dy}{dx} = (2x + 3)^4$.

40. $\dfrac{dy}{dx} = (2 - 3x)^3$.

41. $\dfrac{dy}{dt} = \sqrt{x + 2}\,\dfrac{dx}{dt}$.

42. $\dfrac{dy}{dx} = \sqrt{1 - x}$.

43. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x + 1}}$.

44. $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{3x+2}}$.

45. $\dfrac{dy}{dx} = \sqrt{2-5x}$.

46. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{3-2x^2}}$.

47. $\dfrac{dy}{dx} = \sqrt{(3x+2)^3}$.

48. $\dfrac{dy}{dx} = \sqrt{(3-2x)^3}$.

49. $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2+7}}$.

50. $\dfrac{dy}{dx} = x\sqrt{25-x^2}$.

51. $\dfrac{dy}{dt} = x(x^2+1)^2\dfrac{dx}{dt}$.

52. $\dfrac{dy}{dt} = \dfrac{x}{\sqrt{9-x^2}}\dfrac{dx}{dt}$.

53. Find the equation of the curve whose slope at any point is equal to the square of the abscissa of that point and which passes through the point (3, 4).

54. Find the equation of the curve whose slope at any point is equal to the square root of the abscissa of that point and which passes through the point (4, 8). Sketch the curve.

55. Find the equation of the curve whose slope at any point is equal to the negative reciprocal of the square of the abscissa of that point and which passes through the point (2, 2).

56. The slope of a curve at any point is equal to $(x-3)^{\frac{1}{2}}$, where x is the abscissa of that point. Find the equation of the curve if it passes through the point (7, 5).

57. Find the equation of the curve whose slope at any point is equal to the square of the ordinate of that point and which passes through the point (6, 3).

58. For any value of x the slope of a curve is the negative reciprocal of the slope of $y = \dfrac{x^3}{9}$ for the same value of x. Find the equation of the curve if it passes through the point (3, 1).

59. Show that the curves $x^2 = 4(y+1)$ and $x^2 = -8(y-2)$ intersect at right angles.

60. The radius r, measured in inches, of a circle is increasing at a rate equal to 0.3 times the time t measured in seconds from a certain instant. Find r as a function of t, given that $r = 5$ when $t = 4$.

61. A particle is moving in a straight line. Its distance s, measured in inches, from a fixed point in the line is increasing at a rate equal to $10t$, where t is the time measured in minutes after a certain instant. Express s as a function of t, given that $s = 8$ when $t = 1$.

62. Find the acute angle between the lines drawn tangent to $x^2 = 2y$ and $y^2 = 16x$ at their points of intersection.

63. Water is flowing out through a circular hole in the side and near the bottom of a cylindrical tank 2 feet in diameter. The velocity of the water in the jet, which is 0.5 inch in diameter, is equal to $\sqrt{2gh}$, where h is the height in feet of the surface of the water in the tank above the center of the hole. Find h as a function of the time t (measured in seconds), if $h = 9$ when $t = 0$.

HINT. First show that

$$\frac{\pi\sqrt{2gh}}{2304} = -\pi\frac{dh}{dt},$$

or

$$\frac{1}{\sqrt{h}}\frac{dh}{dt} = -\frac{\sqrt{2g}}{2304}.$$

64. Using the results of Exercise 63, find the time required for the surface of the water in the tank to drop from $h = 25$ feet to $h = 16$ feet. Use $g = 32.2$.

37. Acceleration. The velocity of a body moving in a straight line may be either uniform or it may vary from instant to instant. In the latter case its motion is said to be accelerated, and this applies both to the case where there is an increase in velocity and the case where there is a decrease in velocity.

The time rate of change of the velocity of a body is an important concept in the study of problems in mechanics and in physics. A body falling from a height to the earth is an example of a body moving with a variable velocity. From experience we know that the velocity of a falling body increases with time and hence with the distance through which it has fallen.

If a body is moving in a straight line, and if at the time t its position with respect to a fixed origin on the line is represented by s, the notion of its velocity at a given instant is derived from its average velocity for an interval of time Δt. The average velocity is defined as the ratio of Δs to Δt, where Δs is the change in the displacement corresponding to a change Δt in time. The limiting value of this ratio, as Δt approaches zero, is defined as the velocity of the body at the beginning of the interval Δt. See **§16.**

If the velocity v changes by the amount Δv in an interval of time Δt, the ratio Δv to Δt is called *the average linear acceleration* for the interval of time Δt. *The acceleration at the time t is defined as the limit of the average acceleration as Δt approaches zero. The acceleration is then $\frac{dv}{dt}$, or the time rate of change of the velocity.*[1]

In case of a body falling from a moderate height, it is known experimentally that the time rate of change of its velocity is a constant if the resistance of the air is neglected. This constant is known as the acceleration due to the force of gravity, and is usually represented by the letter g. In the F.P.S. (foot-pound-second) system, g is approximately 32.2. The unit of measure in the F.P.S. system is 1 foot per second per second. Acceleration is the time rate of change of velocity which in turn is the time rate of change of displacement. Hence the unit foot per second per second.

Illustration 1. Study the motion of a body projected vertically upward from the surface of the earth with a speed of 200 feet per second. In this and the following illustrations and exercises the resistance of the air will be neglected.

Let the position of the body be given by the coordinate s, considered to be positive when measured upward. Choose the point from which the body is projected and the instant at which it is projected as the origin of s and t, respectively. From this choice of origin of s and of t, it follows that when $t = 0$, $s = 0$ and $v = 200$.

In this case the differential equation of motion is

$$\frac{dv}{dt} = -g. \qquad (1)$$

It is important that the student should understand clearly the reason for the use of the minus sign in the right-hand member of equation (1). Since the positive sense of s has been chosen upward, a positive velocity will be directed upward. Then a

[1] We suppose here that the body is moving in a straight line. If the path is curved, it will be seen later that the total acceleration is to be thought of as the resultant of two components, one of which produces a change in the direction of the velocity and the other a change in the magnitude of the velocity.

positive acceleration will be associated with an increasing upward velocity, and a negative acceleration, as in the present situation, with a decreasing upward velocity or with an increasing downward velocity.

On integrating equation (1), we obtain

$$v = -gt + C_1.$$

The constant of integration, determined from the condition that $v = 200$ when $t = 0$, is 200. Hence,

$$v = -gt + 200. \tag{2}$$

Replacing v in (2) by $\frac{ds}{dt}$ and integrating, we obtain

$$s = -\tfrac{1}{2}gt^2 + 200t + C_2,$$

where C_2 is zero, since $s = 0$ when $t = 0$. Hence,

$$s = -\tfrac{1}{2}gt^2 + 200t. \tag{3}$$

Equations (2) and (3) give, respectively, the velocity and position of the body at any time.

From (2) we see that v is positive when $t < \frac{200}{g}$; equal to zero when $t = \frac{200}{g}$; and negative when $t > \frac{200}{g}$. Hence the body is moving upward until t reaches the value $\frac{200}{g}$, and is moving downward when t exceeds this value. The body reaches its greatest height when $t = \frac{200}{g}$. Upon substituting this value of t in equation (3) we obtain $s = 621$, approximately, the greatest height in feet to which the body rises.

It must be remembered that s does not necessarily represent the distance through which the body moves, it merely represents the position of the body with respect to the origin of s. This position is represented by a positive or a negative value of the variable s.

If in equation (3) we set s equal to zero, the resulting equation in t has two roots, *viz.*, zero and $\frac{400}{g}$. The first corresponds to the time when the body is projected upward, and the second to the time when it again returns to the earth.

By substituting this second value of t in equation (2) we find that the body returns to the earth with the same speed with which it was thrown upward.

It is to be remembered that in the solution of this problem the resistance of the air was neglected, a factor of considerable importance, especially so for high velocities.

Illustration 2. From a captive balloon, 500 feet above the earth, a body is thrown upward with a speed of 20 feet per second. Study the motion of the body, neglecting the resistance of the air.

Consider s to be positive when measured downward, and choose the origin of s at the balloon and the origin of time t at the instant the body is thrown upward. We then have $s = 0$ and $v = -20$ when $t = 0$.

The differential equation of motion is

$$\frac{dv}{dt} = g. \tag{4}$$

The sign before g is determined by noting that s is considered to be positive when measured downward and using reasoning similar to that used in *Illustration* 1.

On integrating equation (4), we obtain

$$v = gt + C_1.$$

From the condition that $v = -20$ when $t = 0$ it follows that $C_1 = -20$. Hence,

$$v = gt - 20. \tag{5}$$

Substituting $\frac{ds}{dt}$ for v, integrating, and determining the constant of integration so that $s = 0$ when $t = 0$, we obtain

$$s = \tfrac{1}{2}gt^2 - 20t. \tag{6}$$

From equation (5) we see that the body moves upward when $t < \frac{20}{g}$ and downward when $t > \frac{20}{g}\cdot$ The body reaches its greatest height when $t = \frac{20}{g}\cdot$ On substituting this value of t in equation (6) we find the corresponding value of s to be $-\frac{200}{g}\cdot$ Thus at the highest point of its motion the body is $500 + \frac{200}{g}$ feet above the earth.

The student will show that the body reaches the earth in $\frac{20 + 10\sqrt{10g + 4}}{g}$ seconds with a velocity of $10\sqrt{10g + 4}$ feet per second.

Exercises

1. A body is moving in a vertical line according to the law,

$$s = 40 + 120t - 16.1t^2,$$

s being measured upward from the ground. Find velocity and acceleration of the body when $t = 3$. For what values of t is the velocity negative? Positive?

2. The distance of a particle from a fixed point on a straight line on which it is moving is given by the law

$$s = t^3 - 6t^2 - 15t + 7.$$

For what values of t is the particle moving forward? Backward? For what values of t is the velocity increasing? Decreasing?

In each of Exercises 3 to 8, inclusive, the student will begin by writing a differential equation of motion of the moving body. From this equation he will find expressions for velocity and displacement in terms of time. In each case the resistance of the air is to be neglected.

3. A body falls from a captive balloon which is 1000 feet above the ground. How far will the body be from the ground 5 seconds after it leaves the balloon? In how many seconds will the body reach the ground? With what velocity will the body strike the ground?

4. From the balloon of Exercise 3, a body is projected downward with a velocity of 10 feet per second. Answer the questions asked in Exercise 3.

5. From the balloon of Exercise 3, a body is projected upward with a speed of 20 feet per second. Answer the questions asked in Exercise 3. How far will the body be from the ground one second after it is projected upward?

6. A ball is thrown vertically upward with a velocity of 128.8 feet per second. How high will it rise? How long will it take the ball to reach its greatest height? Where will the ball be at the end of the third second? At the end of the seventh second? What will be the velocity of the ball at the end of the sixth second?

7. A balloon is ascending at a rate of 100 feet per second. A stone is let fall from the balloon when it is 1000 feet high. In how many seconds will the stone reach the earth? How far will the stone have traveled in this time? With what velocity will it strike the earth?

8. A body is thrown upward from the ground with a speed of 30 feet per second. One second later a second body is thrown upward with a speed of 40 feet per second. When and where will the two bodies meet? At this instant will the two bodies be moving in the same or in opposite directions?

Hint. Choose as the origin of time the instant the first body is thrown up.

9. The acceleration of a body moving in a straight line is equal to $3t$. Find an expression for s in terms of t, if (a) $s = 0$ and $v = 0$, when $t = 0$; (b) $s = s_0$ and $v = v_0$, when $t = 0$.

10. If the acceleration of a body moving in a straight line is proportional to the time and if $v = v_0$ and $s = s_0$ when $t = 0$, show that $s = \frac{kt^3}{6} + v_0t + s_0$.

CHAPTER IV

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

38. The Derivative of the Product of a Constant and a Variable. Let

$$y = cu,$$

where c is a constant and u a function of x. Let x take on an increment Δx. Then u and y take on the increments Δu and Δy, respectively. Then

$$\begin{aligned} y + \Delta y &= c(u + \Delta u) \\ \Delta y &= c\Delta u \\ \frac{\Delta y}{\Delta x} &= c\,\frac{\Delta u}{\Delta x} \\ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= c \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}. \end{aligned}$$

That is,

$$\frac{dy}{dx} = c\,\frac{du}{dx},$$

or

$$\mathbf{\frac{d(cu)}{dx} = c\,\frac{du}{dx}.}$$

The derivative of the product of a constant and a function is equal to the constant times the derivative of the function.

Illustrations.

1. $\dfrac{d(3x^2)}{dx} = 3\dfrac{d(x^2)}{dx} = 6x.$
2. $\dfrac{d[4(x-2)^2]}{dx} = 4\dfrac{d(x-2)^2}{dx} = 8(x-2).$

3. $$\frac{d[-\frac{2}{3}(x^2-5)^{\frac{1}{2}}]}{dx} = -\frac{2}{3}\frac{d(x^2-5)^{\frac{1}{2}}}{dx}$$
$$= -\frac{2}{3}\,\frac{1}{2}(x^2-5)^{-\frac{1}{2}}\frac{d(x^2-5)}{dx}$$
$$= -\frac{2}{3}\frac{x}{(x^2-5)^{\frac{1}{2}}}.$$

39. The Derivative of the Product of Two Functions. Let

$$y = uv,$$

where u and v are functions of x.

Let x take on an increment Δx. Then u and v take on the increments Δu and Δv, respectively, and y takes on the increment Δy.

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$
$$y + \Delta y = uv + u\Delta v + v\Delta u + \Delta u\Delta v$$
$$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v$$
$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}.$$
$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} u\frac{\Delta v}{\Delta x} + \lim_{\Delta x\to 0} v\frac{\Delta u}{\Delta x} + \lim_{\Delta x\to 0}\Delta u\cdot\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}.$$

The last term disappears since Δu approaches zero when Δx approaches zero.

Hence,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

or

$$\mathbf{\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.} \tag{1}$$

The derivative of the product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first.

Illustrations.

1. $$\frac{d(x+2)(x+3)}{dx} = (x+2)\frac{d(x+3)}{dx} + (x+3)\frac{d(x+2)}{dx}$$
$$= (x+2) + (x+3) = 2x + 5.$$

CHAPTER IV

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

38. The Derivative of the Product of a Constant and a Variable. Let

$$y = cu,$$

where c is a constant and u a function of x. Let x take on an increment Δx. Then u and y take on the increments Δu and Δy, respectively. Then

$$\begin{aligned} y + \Delta y &= c(u + \Delta u) \\ \Delta y &= c\Delta u \\ \frac{\Delta y}{\Delta x} &= c\,\frac{\Delta u}{\Delta x} \\ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= c \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}. \end{aligned}$$

That is,

$$\frac{dy}{dx} = c\,\frac{du}{dx},$$

or

$$\mathbf{\frac{d(cu)}{dx} = c\,\frac{du}{dx}.}$$

The derivative of the product of a constant and a function is equal to the constant times the derivative of the function.

Illustrations.

1. $\dfrac{d(3x^2)}{dx} = 3\dfrac{d(x^2)}{dx} = 6x.$
2. $\dfrac{d[4(x-2)^2]}{dx} = 4\dfrac{d(x-2)^2}{dx} = 8(x-2).$

3. $$\frac{d[-\frac{2}{3}(x^2-5)^{\frac{1}{2}}]}{dx} = -\frac{2}{3}\frac{d(x^2-5)^{\frac{1}{2}}}{dx}$$
$$= -\frac{2}{3}\,\frac{1}{2}(x^2-5)^{-\frac{1}{2}}\,\frac{d(x^2-5)}{dx}$$
$$= -\frac{2}{3}\frac{x}{(x^2-5)^{\frac{1}{2}}}.$$

39. The Derivative of the Product of Two Functions. Let

$$y = uv,$$

where u and v are functions of x.

Let x take on an increment Δx. Then u and v take on the increments Δu and Δv, respectively, and y takes on the increment Δy.

$$\begin{aligned} y + \Delta y &= (u + \Delta u)(v + \Delta v) \\ y + \Delta y &= uv + u\Delta v + v\Delta u + \Delta u\Delta v \\ \Delta y &= u\Delta v + v\Delta u + \Delta u\Delta v \\ \frac{\Delta y}{\Delta x} &= u\,\frac{\Delta v}{\Delta x} + v\,\frac{\Delta u}{\Delta x} + \Delta u\,\frac{\Delta v}{\Delta x}. \end{aligned}$$

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} u\frac{\Delta v}{\Delta x} + \lim_{\Delta x\to 0} v\frac{\Delta u}{\Delta x} + \lim_{\Delta x\to 0}\Delta u \cdot \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}.$$

The last term disappears since Δu approaches zero when Δx approaches zero.

Hence,

$$\frac{dy}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx},$$

or

$$\frac{\mathbf{d(uv)}}{\mathbf{dx}} = \mathbf{u}\,\frac{\mathbf{dv}}{\mathbf{dx}} + \mathbf{v}\,\frac{\mathbf{du}}{\mathbf{dx}}. \tag{1}$$

The derivative of the product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first.

Illustrations.

1. $$\frac{d(x+2)(x+3)}{dx} = (x+2)\,\frac{d(x+3)}{dx} + (x+3)\,\frac{d(x+2)}{dx}$$
$$= (x+2) + (x+3) = 2x + 5.$$

2. $$\frac{d(x^2+3x)(x-2)}{dx} = (x^2+3x)\frac{d(x-2)}{dx} + (x-2)\frac{d(x^2+3x)}{dx}$$
$$= (x^2+3x) + (x-2)(2x+3)$$
$$= 3x^2 + 2x - 6.$$

3. If $x^2 + xy^3 + y = 10$,

$$2x + x\frac{d(y^3)}{dx} + y^3 + \frac{dy}{dx} = 0,$$

$$2x + 3xy^2\frac{dy}{dx} + y^3 + \frac{dy}{dx} = 0.$$

Whence

$$\frac{dy}{dx} = -\frac{2x+y^3}{3xy^2+1}.$$

4. Given $xy = 10$ where x and y are functions of t. Then

$$x\frac{dy}{dt} + y\frac{dx}{dt} = 0.$$

Hence

$$\frac{dy}{dt} = -\frac{y}{x}\frac{dx}{dt}.$$

Exercises

Find $f'(x)$ (see §**18**):

1. $f(x) = (x^2 - 3x)(2x + 5)$.
2. $f(x) = (3x - 2)(5 - 2x)$.
3. $f(x) = (x + 1)^2(x^2 - 5)$.
4. $f(x) = x(x + 2)^5$.
5. $f(x) = x\sqrt{x + 1}$.
6. $f(x) = (x - 2)\sqrt{x}$.
7. $f(x) = (2x - 3)\sqrt{x^2 + 3}$.
8. $f(x) = (x + 2)\sqrt{9 - x^2}$.
9. $f(x) = x\sqrt{5 + x^2}$.
10. $f(x) = (5 - 2x)\sqrt{3 - x^2}$.

Find $\frac{dy}{dx}$:

11. $y = (x + 1)(2x - 1)^{\frac{3}{2}}$.
12. $y = (2 - x)\sqrt{3x + 2}$.
13. $y = (3x - 2)\sqrt{x}$.
14. $y = x\sqrt{4 - 5x - 3x^2}$.
15. $x^2y^2 + 3x^3 - 7y^3 = 10$.
16. $x^2y^3 - 3xy^2 = 25$.
17. $xy = 20$.
18. $x^2y = 10$.
19. $2x\sqrt{y} = 11$.
20. $5x^3y^3 - x^2 + 7y = 5$.

Find $\frac{dy}{dt}$:

21. $x^2y = 15$.
22. $xy^2 = 10$.

Find $\frac{dp}{dt}$:

23. $pv = C.$ **24.** $pv^{1.4} = C.$

40. The Derivative of the Quotient of Two Functions. Let

$$y = \frac{u}{v},$$

where u and v are functions of x. Then

$$yv = u.$$

Differentiating by the rule of §39

$$y\frac{dv}{dx} + v\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{\frac{du}{dx} - y\frac{dv}{dx}}{v}.$$

Replacing y by its value, $\frac{u}{v}$,

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}. \qquad (1)$$

The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

The student should, as an exercise, derive formula (1) by the method of increments.

Illustration:

$$\frac{d\left(\frac{x^2+1}{x-2}\right)}{dx} = \frac{(x-2)\frac{d(x^2+1)}{dx} - (x^2+1)\frac{d(x-2)}{dx}}{(x-2)^2}$$

$$\frac{(x-2)(2x) - (x^2+1)}{(x-2)^2}$$

$$= \frac{x^2-4x-1}{(x-2)^2}.$$

Exercises

Find $f'(x)$:

1. $f(x) = \dfrac{x-1}{x+2}$.

2. $f(x) = \dfrac{x^2+1}{x^2-9}$.

3. $f(x) = \dfrac{2x+3}{3x-4}$.

4. $f(x) = \dfrac{x+3}{2-x}$.

Find $\dfrac{dy}{dx}$:

5. $y = \dfrac{x^2+3}{2-x^2}$.

6. $y = \dfrac{x}{1+x^2}$.

7. $y = \dfrac{x+1}{\sqrt{x+2}}$.

8. $y = \dfrac{\sqrt{1-x}}{1+x^2}$.

9. $y = \dfrac{x^2+2}{\sqrt{1-x^2}}$.

10. $y = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$.

11. $y = \dfrac{\sqrt{x}}{1+x^2}$.

12. $y = \dfrac{\sqrt{x-3}}{x+2}$.

13. $y = \dfrac{x+5}{\sqrt{9-x^2}}$.

14. $y = \dfrac{2x-3}{\sqrt{3x+5}}$.

15. $y = \dfrac{4-3x}{\sqrt{5-3x^2}}$.

41. Maximum and Minimum Values of a Function. In Chapter I it was shown that the derivative of a function with respect to its argument is equal to the slope of the tangent drawn to the curve representing the function. The derivative is positive where the function is increasing and negative where the function is decreasing. These facts enable us to determine the maximum and minimum values of a function.

Additional exercises in finding maximum and minimum values of a function will be given in this section.

Illustration 1. Let

$$y = 2x^3 + 3x^2 - 12x - 10.$$

$$\frac{dy}{dx} = 6x^2 + 6x - 12 = 6(x+2)(x-1)$$

If x is less than -2, both factors of the derivative are negative. Then for all values of x less than -2, the derivative is positive and the function is increasing. If x is greater than -2 and less than 1, the first factor of the derivative is positive and the second factor is negative. Hence, if $-2 < x < 1$, the derivative is negative and the function is decreasing. If x is greater than 1, the derivative is positive and the function is again increasing.

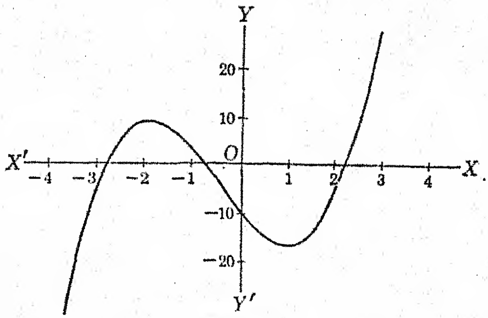

FIG. 22.

The function changes from an increasing to a decreasing function when x passes through the value -2, and changes from a decreasing to an increasing function when x passes through the value 1. Hence the function has a maximum value when x equals -2, and a minimum value when x equals 1. By substituting -2 and 1 for x, the corresponding values of y are found to be 10 and -17, respectively (see Fig. 22). The more important results of the above discussion are put in tabular form below.

x	$x+2$	$x-1$	$\dfrac{dy}{dx}$	Function
$x < -2$	$-$	$-$	$+$	Increasing.
$-2 < x < 1$	$+$	$-$	$-$	Decreasing.
$1 < x$	$+$	$+$	$+$	Increasing.
$x = -2$	0	$-$	0	Max. value = 10.
$x = 1$	$+$	0	0	Min. value = -17.

It is to be observed that -2 and $+1$ are the only values of x at which the derivative can change sign and that these are the values that need to be examined in finding the maximum and minimum values of the function.

Illustration 2. The strength of a rectangular beam varies as its breadth and the square of its depth. Find the dimensions of the strongest beam that can be cut from a circular log 20 inches in diameter.

Denoting by x, y, and s the breadth, depth, and strength, respectively, of the beam, it follows that

$$s = kxy^2 = kx(400 - x^2),$$

and

$$\frac{ds}{dx} = k(400 - 3x^2),$$

from which we see that s has a maximum value when

$$x = \frac{20}{\sqrt{3}} = \tfrac{20}{3}\sqrt{3} = 11.55.$$

Then to find the depth of the strongest beam, substitute this value of x in $y = \sqrt{400 - x^2}$. Thus

$$y = \sqrt{400 - \tfrac{400}{3}} = 20\sqrt{1 - \tfrac{1}{3}} = \tfrac{20}{3}\sqrt{6} = 16.33.$$

The breadth and depth of the strongest beam are 11.55 and 16.33 inches, respectively.

Exercises

In each of Exercises 1 to 17, inclusive, find for what values of x the function is increasing; decreasing. Find the maximum and minimum values if there are any, and sketch a curve representing the function.

1. $y = 2x^3 - 3x^2 + 6.$

2. $y = -x^3 - 3x^2 + 2.$

3. $y = x^3 - 3x + 7.$

4. $y = x^3 - 3x^2 - 9x + 5.$

5. $y = x^2(x^2 - 1).$

6. $y = x(x - 1)^3.$

7. $y = x^2(x - 1)^2.$

8. $y = \dfrac{x}{1 + x^2}.$

9. $y = x + \sqrt{8 - x^2}.$

10. $y = \dfrac{\sqrt{x}}{1 + x}.$

11. $y = \dfrac{1}{x^2 + 1}$.

12. $y = \dfrac{x}{x^2 - 1}$.

13. $y = \dfrac{x}{x^3 + 2}$.

14. $y = \dfrac{x^2}{x^2 - 1}$.

15. $y = \dfrac{(x - 1)^2}{4(x + 1)}$.

16. $y = \dfrac{x + 1}{x^2 - 3x}$.

17. $y = \dfrac{x^3}{(x^2 - 1)^2}$.

18. Find the dimensions of the largest rectangle that can be inscribed in a circle of radius a.

19. What positive number exceeds its cube by the greatest amount?

20. A V-shaped trough of maximum capacity is to be made from two boards 10 inches wide. Find the width of the trough at the top.

21. Find the dimensions of the greatest rectangular field that can be enclosed by 500 rods of fence.

22. Equal squares are cut from the corners of a rectangular piece of tin 42 by 24 inches. The rectangular projections are then turned up to form a tray. Find the size of the squares to be cut out in order that the tray may have the greatest volume.

23. Find the dimensions of the largest rectangular field that can be enclosed with 500 rods of fence, if the field adjoins a railroad right of way along which a fence is already constructed.

24. Find the dimensions of the rectangle of maximum perimeter that can be inscribed in a given circle of radius a.

25. The stiffness of a rectangular beam varies as its breadth and as the cube of its depth. Find the dimensions of the stiffest beam that can be cut from a circular log 16 inches in diameter.

26. A cylindrical cistern is to be constructed with open top and of a given capacity. Find ratio of the diameter to the depth of the cistern of minimum total surface (sides and bottom).

27. A ship A is 40 miles directly north of a ship B, at a certain instant. Ship B is sailing due east at the rate of 10 miles per hour, and ship A is sailing due south at the rate of 15 miles per hour. Show that the distance between the ships is expressed by

$$s = \sqrt{325t^2 - 1200t + 1600},$$

where t is the number of hours since A was due north of B. At what time are the ships nearest together? At what rate are they approaching or separating when $t = 1$? When $t = 2$? When $t = 5$?

28. A rectangle is inscribed in a right triangle whose sides are 6, 8, and 10 feet respectively. Find the dimensions of the rectangle of maximum area if its base lies on the hypotenuse of the triangle.

42. Derivative of a Function of a Function. If $y = \phi(u)$ and $u = f(x)$, y is a function of x. The derivative of y with respect to x can be found without eliminating u. For any set of corresponding increments, Δx, Δy, and Δu,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}.$$

Hence,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}.$$

Since Δu approaches zero as Δx approaches zero,

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{dy}}{\mathbf{du}}\frac{\mathbf{du}}{\mathbf{dx}}. \tag{1}$$

Illustration. *Let*

$$y = u^3 + 5$$

and

$$u = 3x^2 + 7x + 10.$$

$$\frac{dy}{du} = 3u^2$$

and

$$\frac{du}{dx} = 6x + 7.$$

Then

$$\begin{aligned}\frac{dy}{dx} &= 3u^2(6x + 7)\\ &= 3(3x^2 + 7x + 10)^2(6x + 7).\end{aligned}$$

43. Inverse Functions. If x is given as a function of y, say $x = \phi(y)$, $\frac{dy}{dx}$ can be found by the rule

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},$$

which is easily proved.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\frac{\Delta x}{\Delta y}} = \frac{1}{\frac{dx}{dy}}.$$

Illustration. If $x = 5y^6 + 7y^2 + 3$,

$$\frac{dx}{dy} = 30y^5 + 14y$$

and

$$\frac{dy}{dx} = \frac{1}{2y(15y^4 + 7)}.$$

Exercises

1. Find $\frac{dy}{dx}$ in terms of x, if:

(*a*) $y = 3u^2 + 2u$ and $u = x^2 - 5$.
(*b*) $y = \sqrt{9 - u^2}$ $\quad u = 2x - 3$.
(*c*) $y = u^4$ $\quad u = x^2 - 3x + 4$.
(*d*) $y = \frac{1}{\sqrt{u^2 + 3}}$ $\quad u = x^2 + 3$.

2. Find $\frac{dy}{dx}$:

(*a*) $x = \sqrt{y^2 + 7}$.
(*b*) $x = \frac{1}{\sqrt{3y^2 + 2}}$.
(*c*) $x = \frac{1}{(y^2 - 3)^2}$.
(*d*) $x = y^3 + 4y^2 - 2y + 5$.
(*e*) $x = y + \sqrt{9 - y^2}$.
(*f*) $x = \frac{y + 2}{y - 1}$.
(*g*) $x = \frac{y - 3}{\sqrt{y + 1}}$.
(*h*) $x = y\sqrt{4 - y^2}$.

44. Parametric Equations. If the equation of a curve is given in parametric form, $x = f(t)$, $y = \phi(t)$, it is important to be able to find the derivative of y with respect to x without eliminating t between the given equations. A rule for doing this can be derived.

If t is given an increment Δt, x and y take on the increments Δx and Δy, respectively. Then

$$\frac{\Delta y}{\Delta x} = \frac{\dfrac{\Delta y}{\Delta t}}{\dfrac{\Delta x}{\Delta t}},$$

and

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta x} = \frac{\lim\limits_{\Delta t \to 0} \dfrac{\Delta y}{\Delta t}}{\lim\limits_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t}},$$

or

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}.$$

Exercises

The student should sketch the curves in Exercises 1 to 4.

1. Find the slope at (4,1) of the curve whose parametric equations are

$$x = t^2$$
$$y = t - 1.$$

Find $\dfrac{dy}{dx}$:

2. $x = t^2 + 1$
$y = t^3.$

3. $x = 2t + 3$
$y = t^3.$

4. $x = t^2 + 4$
$y = 3t - 1.$

5. $x = \dfrac{1}{2t - 3}$
$y = t^2 - 5.$

45. Lengths of Tangent, Normal, Subtangent, and Subnormal. In Fig. 23, PT is the tangent and PN is the normal at P. The lengths of the lines PT, PN, TD, and DN are called the tangent,

the normal, the subtangent, and the subnormal, respectively, for the point P. Show that the lengths of these lines are:

$$TD = \frac{y}{\frac{dy}{dx}}, \tag{1}$$

$$DN = y\frac{dy}{dx}, \tag{2}$$

$$PT = \frac{y}{\frac{dy}{dx}}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \tag{3}$$

$$PN = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{4}$$

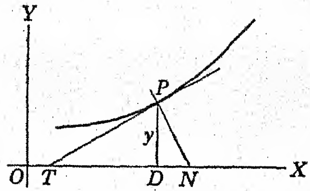

FIG. 23.

Exercises

1. Find the length of the tangent, the normal, the subtangent, and the subnormal at the point (3, 9) on the curve $y = x^2$.

2. Find the length of the tangent, the normal, the subtangent, and the subnormal at the point (4, 6) on the curve $y^2 = 9x$.

3. Show that the length of the subnormal for points on the curve $y^2 = 2px$ is constant.

4. Show, by two methods, that the vertex of the parabola $y^2 = 2px$ bisects the subtangent corresponding to any point on the curve.

5. Find the length of the subtangent at the point (3, 4) on the circle $x^2 + y^2 = 25$.

6. Find the length of the subtangent at the point (3, 2) on the ellipse $x^2 + 4y^2 = 25$. Compare your result with that of the preceding exercise.

7. Find the length of the subtangent at a point on the curve $xy = k$.

8. Find the equation of the curve through the point (2, 5) whose subnormal is of constant length 4.

9. Find the equation of the curve through the point (2, 6) whose subtangent at any point is equal to the square of the ordinate of that point.

10. Find the equation of the curve passing through (2, 9) whose subtangent at any point is equal to the square root of the ordinate of that point.

11. Find the equation of the curve passing through (2, 1) whose subnormal at any point is equal to the product of the coordinates of that point.

12. Find the equation of the curve passing through (−2, 3) whose subnormal at any point is equal to the square of the abscissa of that point.

Miscellaneous Exercises

Differentiate the following 20 expressions with respect to x:

1. $(2 - x)^3$.

2. $\sqrt{3 - x^2}$.

3. $\dfrac{4}{(x-3)^3}$.

4. $x\sqrt{1 - 3x^2}$.

5. $\dfrac{x+3}{\sqrt{x+2}}$.

6. $(5x^2 + 2)\sqrt{3 - x^2}$.

7. $x^{\frac{1}{3}} - x^{\frac{1}{2}} + 3$.

8. $(2x^2 - 3)^{\frac{1}{2}}$.

9. $(x - 1)^{\frac{1}{2}}(2 - x)^{\frac{1}{3}}$.

10. $\dfrac{(x-1)^{\frac{1}{2}}}{(2-x)^{\frac{1}{3}}}$.

11. $(2x + 3)(x^2 + 4)$.

12. $x^2\sqrt{x^2 - 9}$.

13. $\dfrac{4}{5 - x^2}$.

14. $(x^2 + 2)\sqrt{x}$.

15. $\dfrac{\sqrt{9 - x^2}}{x}$.

16. $\sqrt{2x^3 - 5x^2 + 11}$.

17. $(2 - x)^2(3x^2 - 1)$.

18. $(3x^3 - 2x^2 - 3)^{\frac{3}{2}}$.

19. $(1 - x^2)^{\frac{2}{3}}(x - x^2)^{-\frac{1}{2}}$.

20. $\dfrac{(1 - x^2)^{\frac{2}{3}}}{(x - x^2)^{\frac{3}{2}}}$.

Integrate the following 20 expressions:

21. $\dfrac{dy}{dx} = x(1 - 3x^2)^3$.

22. $\dfrac{dy}{dx} = x\sqrt{2 - x^2}$.

23. $\dfrac{dy}{dx} = \dfrac{x}{(1 - 3x^2)^2}$.

24. $\dfrac{dy}{dx} = \dfrac{2 - x}{\sqrt{4x - x^2}}$.

25. $\dfrac{dy}{dx} = \dfrac{x^2}{\sqrt{5x^3 - 3}}$.

26. $\dfrac{dy}{dx} = (2x^2 - 4x)^2(x - 1)$.

27. $\dfrac{dy}{dx} = \dfrac{2}{(1 - 3x)^2}$.

28. $\dfrac{dy}{dx} = (3 + x^2)^2$.

29. $\dfrac{dy}{dx} = \dfrac{(3 - \sqrt{x})^2}{\sqrt{x}}$.

30. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2x - 3}}$.

31. $\dfrac{dy}{dx} = (2x^{\frac{1}{2}} - 3x^{\frac{1}{3}})^2$.

32. $\dfrac{dy}{dx} = x(x - 2x^2)^2$.

33. $\dfrac{dy}{dx} = \dfrac{x^2 - 2x + 2}{(x^3 - 3x^2 + 6x)^3}$.

34. $\dfrac{dy}{dx} = \dfrac{2}{(1 - x)^5}$.

35. $\dfrac{dy}{dx} = \dfrac{3}{x^{\frac{1}{3}}} - \dfrac{2}{x^{\frac{1}{2}}}$.

36. $\dfrac{dy}{dx} = x(1 - x^2)^{\frac{1}{2}}$.

37. $\dfrac{dy}{dx} = (a^{\frac{1}{2}} - x^{\frac{1}{2}})^2$.

38. $\dfrac{dy}{dx} = (2 + \sqrt{x})^3$.

39. $\dfrac{dy}{dx} = \dfrac{1 - x}{(x^2 - 2x)^{\frac{3}{2}}}$.

40. $\dfrac{dy}{dx} = \dfrac{x - 1}{x^3}$.

Find $\dfrac{dy}{dx}$ in each of the following four exercises:

41. $2x^2 - y^2 = a^2$.

42. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.

43. $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$.

44. $x^3y^2 + 3xy^3 = 13$.

45. A ladder 20 feet long leans against the vertical wall of a building. If the lower end of the ladder is drawn out along the horizontal ground at the rate of 2 feet per second, at what rate is its upper end moving down when the lower end is 10 feet from the wall?

HINT. Let AC, Fig. 24, be the wall and let CB be the ladder. Let $AB = x$ and $AC = y$. Then

$$y = \sqrt{400 - x^2}$$

and

$$\frac{dy}{dt} = \frac{-x}{\sqrt{400 - x^2}} \frac{dx}{dt}.$$

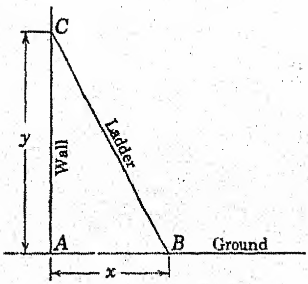

FIG. 24.

But, since $\dfrac{dx}{dt} = 2$,

$$\frac{dy}{dt} = \frac{-2x}{\sqrt{400 - x^2}}.$$

The negative sign of the derivative indicates that the upper end of the ladder is moving down.

46. Answer the question of Exercise 45, if $x = 0$; $x = 2$; $x = 15$; $x = 20$.

47. With the statement of Exercise 45, find the rate at which the area of the triangle ABC, Fig. 24, is increasing when the lower end of the ladder is 5 feet from the wall.

48. With the statement of Exercise 45, find the position of the ladder when the area of the triangle ABC, Fig. 24, is a maximum.

49. The two legs of an isosceles triangle are each 15 inches long. The base of the triangle is increasing at the rate of 0.1 inch per minute. At what rate is the area of the triangle increasing when the altitude is 8 inches?

50. The radius of the base of a cylinder is increasing at the rate of 0.5 foot per minute and the altitude at the rate of 0.2 foot per minute. At what rate is the volume of the cylinder increasing when the radius is 2 feet and the altitude 5 feet?

51. A gas in a cylindrical vessel is being compressed by a moving piston. The conditions are such that Boyle's law, $pv = C$, is satisfied. If at a certain instant the volume is decreasing at the rate of 1.2 cubic feet per second, at what rate is the pressure changing, if at this instant the pressure is 4000 pounds per square foot and the volume is 12 cubic feet?

52. A point is moving along the curve $y = x^2 - 4x$. Find the point on the curve at which the ordinate is changing four times as fast as the abscissa.

53. The ordinate of a point moving on a curve is changing at a rate equal to twice the rate of change of the abscissa. Find the equation of the curve if it passes through the point $(2, 3)$.

54. Show that the curves $\frac{x^2}{49} + \frac{y^2}{25} = 1$ and $\frac{x^2}{16} - \frac{y^2}{8} = 1$ intersect at right angles.

55. Show that the curves $y^2 - 2x - 1 = 0$ and $y^2 + 4x - 4 = 0$ intersect at right angles.

56. Show that the curves $y^2 - 2px - p^2 = 0$ and $y^2 + 2kx - k^2 = 0$ intersect at right angles, if p and k are both positive.

57. A point moves so that the ratio of the rate of change of its ordinate to the rate of change of its abscissa is three times its abscissa. Find the equation of the locus of the point if the locus passes through the point $(-2, 3)$.

58. Same as Exercise 57, but let the ratio be equal to the square root of the ordinate.

59. The legs of an isosceles triangle are each k units in length. Find the length of the base when the area of the triangle is a maximum.

60. A man has N rods of fencing with which he wishes to enclose a rectangular field and divide it into two equal parts by a fence running

parallel to one of the sides of the rectangle. Find the ratio of the length to the breadth of the rectangle in order that the area of the field may be a maximum.

61. At a certain instant the pressure in a vessel containing air is 3600 pounds per square foot and the volume is 10 cubic feet. The volume is increasing at the rate of 2.3 cubic feet per second. At what rate is the pressure changing, if the adiabatic law $pv^{1.4} = C$ is satisfied?

62. A gas in a cylindrical vessel 10 inches in diameter is being compressed by a piston. If Boyle's law, $pv = C$, is satisfied and the piston is moving at the rate of 3.7 inches per minute, find the rate at which the pressure is changing, when the pressure is 50 pounds per square inch and the volume is 5 cubic feet.

63. A captive balloon is 4000 feet high. A ball is projected vertically downward from the balloon with an initial velocity of 120 feet per second. The altitude of the sun being 30°, express the velocity of the shadow of the ball along the ground as a function of the time t. Find this velocity at the instant the ball leaves the balloon. At the instant it is half way to the ground. At the instant it strikes the ground.

64. Two railroad tracks AOB and MON intersect at O, making the angle NOB equal to 60°. A train is running along AB at the rate of 30 miles per hour and passes through O at 11 A.M. A second train runs along MN at the rate of 50 miles per hour and passes through O at 1 P.M. How fast are the trains approaching or separating at 10 A.M.? At 2 P.M.? When are they nearest together?

65. The base of an isosceles triangle is 20 inches. The angle at the vertex is 40°. Find the dimensions of the maximum inscribed rectangle whose base rests on the base of the triangle.

66. Water is flowing from an orifice in the side of a cylindrical tank whose cross section is 30 square feet in area. The velocity of the water in the jet is equal to $\sqrt{2gh}$, where h is the height in feet of the surface of the water above the orifice. If the cross section of the jet is 0.005 square foot, how long will it take the water to fall from a height 9 feet to a height of 4 feet above the orifice?

67. Find the equation of the straight line tangent to

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

at the point (3, 2.4). (Use implicit differentiation.)

68. Find the equation of the straight line tangent to $y^2 = 4x$ at the point (1, 2).

69. Find the equation of the straight line normal to $y^2 = 4x$ at the point (1, 2).

70. Find the equation of the straight line tangent to $x^2 + y^2 = 25$ at the point (4, 3).

71. Find the equation of the line tangent to the parabola $y^2 = 2px$ at the point (x_1, y_1). *Ans.* $yy_1 = p(x + x_1)$.

72. Find the equation of the line tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at the point (x_1, y_1). *Ans.* $b^2xx_1 + a^2yy_1 = a^2b^2$.

73. Find the equation of the line tangent to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ at the point (x_1, y_1). *Ans.* $b^2xx_1 - a^2yy_1 = a^2b^2$.

74. Find the equation of a straight line passing through (2, 3) and forming in the first quadrant with the coordinate axes, a triangle of minimum area.

75. A wire of length k is cut into two parts. One part is bent to form a square and the other part a circle. Find the ratio of the length of a side of the square to the length of the diameter of the circle when the combined area of the square and the circle is a maximum.

CHAPTER V

SECOND DERIVATIVE. POINT OF INFLECTION. MAXIMA AND MINIMA

46. Second Derivative, Concavity. Since the first derivative of a function of x is itself a function of x, we can take the derivative of the first derivative. *The derivative of the first derivative is called the second derivative.* If y is a function of x, say, $y = f(x)$, the second derivative is represented by the symbol $\frac{d\left(\frac{dy}{dx}\right)}{dx}$, or $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, or more commonly by the symbol $\frac{d^2y}{dx^2}$, which is read "the second derivative of y with respect to x."

The symbols y'' and $f''(x)$ are also used to denote the second derivative.

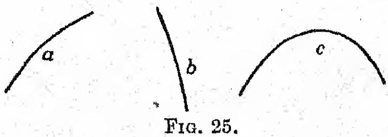

FIG. 25.

The derivative of the second derivative is called the *third derivative.* It is designated by $\frac{d^3y}{dx^3}$, or if $y = f(x)$ by $f'''(x)$. The nth derivative is designated by $\frac{d^ny}{dx^n}$, or by $f^{(n)}(x)$.

Between the points A and C, Fig. 14, where the curve is concave downward, the slope of the tangent decreases from large positive values near A to negative values near C. This means that the tangent revolves in a clockwise direction as the point of tangency moves along the curve from A toward C. Clearly, this will always happen for any portion of a curve that is concave

downward (see Fig. 25, a, b, and c). The slope decreases as the point of tangency moves to the right.

On the other hand, if a portion of a curve is concave upward, the slope of the tangent increases as the point of tangency moves to the right. Thus in Fig. 14, the slope of the tangent is negative at C and increases steadily to positive values at E. The same thing is evidently true for any portion of a curve that is concave upward. In this case the tangent line revolves in a counterclockwise direction.

Since the first derivative of a function is equal to the slope of the tangent to the curve representing the function, what has just been said can be stated concisely as follows:

If an arc of a curve is concave upward, the first derivative is an increasing function, while if the curve is concave downward, the first derivative is a decreasing function.

If the second derivative of a function is positive between certain values of the independent variable x, the first derivative is an increasing function, the tangent line revolves in a counterclockwise direction, and consequently the curve representing the function is concave upward between the values of x in question. If the second derivative is negative, the first derivative is a decreasing function and the curve is concave downward. Thus in Fig. 14, the second derivative is negative between A and C and between E and G. It is positive between C and E, and between G and I.

47. Points of Inflection. Points at which a curve ceases to be concave downward and becomes concave upward, or *vice versa*, are called *points of inflection*.

At such points the second derivative changes sign. C, E, and G, Fig. 14, are points of inflection. At C, for instance, the second derivative changes from negative values to positive values.

Illustration 1. Study the curve $y = \frac{1}{6}x^3$ by means of its derivatives.

Differentiating,

$$\frac{dy}{dx} = \frac{1}{2}x^2,$$

$$\frac{d^2y}{dx^2} = x.$$

When $x < 0$, $\frac{d^2y}{dx^2} < 0$, $\frac{dy}{dx} = \frac{1}{2}x^2$ is a decreasing function, and the curve $y = \frac{1}{6}x^3$ is concave downward. When $x > 0$, $\frac{d^2y}{dx^2} > 0$, $\frac{dy}{dx}$ is an increasing function, and the curve $y = \frac{1}{6}x^3$ is concave upward.

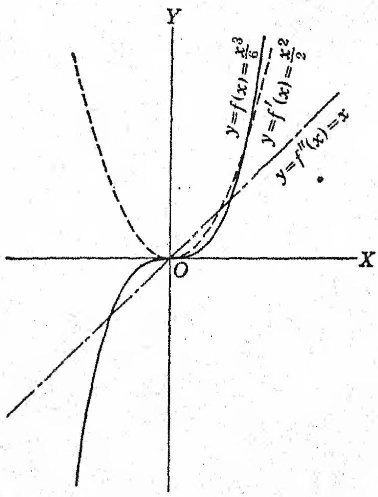

Fig. 26.

At the point where $x = 0$, $\frac{d^2y}{dx^2}$ changes sign from minus to plus and the curve changes from being concave downward to being concave upward. Hence (0, 0) is a point of inflection.

Since $\frac{dy}{dx}$ is positive except when $x = 0$, $y = \frac{1}{6}x^3$ is an increasing function except when $x = 0$. When $x = 0$, the curve has a horizontal tangent.

In Fig. 26 the graphs of the function $y = \frac{1}{6}x^3$ and of its first and second derivatives are drawn on the same axes. Trace out in this figure the properties mentioned in the discussion.

Illustration 2. Let

$$y = \tfrac{1}{6}x^3 - x^2 + \tfrac{3}{2}x + 2.$$

Differentiating,

$$\frac{dy}{dx} = \tfrac{1}{2}x^2 - 2x + \tfrac{3}{2}$$

$$= \tfrac{1}{2}(x - 1)(x - 3).$$

$$\frac{d^2y}{dx^2} = x - 2.$$

At $x = 2$, $\frac{d^2y}{dx^2}$ changes sign from minus to plus. Hence the curve is concave downward to the left, and concave upward to the right

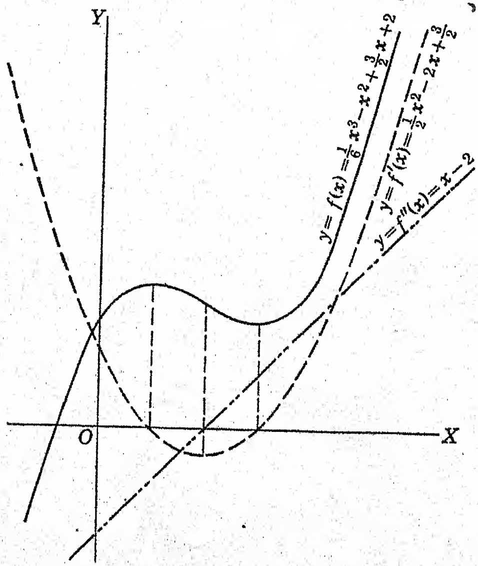

FIG. 27.

of the line $x = 2$. The point on the curve whose abscissa is 2 is then a point of inflection. The value of the function corresponding to $x = 1$ is a maximum value, and the value of the function corresponding to $x = 3$ is a minimum value. See Fig. 27 for a

sketch of the function and its first and second derivatives. Trace out in the figure what has been given in the discussion.

The more important facts concerning the graph of this function are put in tabular form below.

x	$\frac{d^2y}{dx^2}$	$\frac{dy}{dx}$	Curve
$x < 2$	$-$	Decreasing	Concave downward.
$x > 2$	$+$	Increasing	Concave upward.
$x = 2$	0		Point of inflection $(2, 2\frac{1}{3})$.
$x < 1$		$+$	Rising.
$1 < x < 3$		$-$	Falling.
$x > 3$		$+$	Rising.
$x = 1$		0	Maximum point $(1, 2\frac{2}{3})$.
$x = 3$		0	Minimum point $(3, 2)$.

Exercises

Find the maximum and minimum points and points of inflection of the following curves. Sketch the curves.

1. $2y = 3x^2 - x^3$.
2. $2y = x^3 + 3x^2$.
3. $2y = -(3x^2 + x^3)$.
4. $3y = 4x^3 + 9x^2$.
5. $2y = x^3 - 3x + 2$.
6. $2y = x^3 - 3x^2 + 6$.
7. $y = x^4 - 2x^3$.
8. $y = x^4 + 2x^3 - 2x$.
9. $xy = x^2 + 1$.
10. $xy = x^2 + 3x - y + 3$.
11. $4y = x^2(x - 4)^2$.
12. $y = x^3 + ax + b$, $a < 0$.
13. $y = \dfrac{4a^3}{x^2 + a^2}$.
14. $y = \dfrac{12x}{x^2 + 9}$.

48. Second Derivative Test for Maxima and Minima. Let us consider again *Illustration* 2 of the preceding section. The first and second derivatives are

$$y' = \tfrac{1}{2}(x - 1)(x - 3) \tag{1}$$

and

$$y'' = x - 2. \tag{2}$$

In this case the first derivative changes sign from plus to minus at the point $(1, 2\frac{2}{3})$, and from minus to plus at the point $(3, 2)$ by passing through the value zero. In the vicinity of the maximum point $(1, 2\frac{2}{3})$ the curve is concave downward, and in the vicinity of the minimum point $(3, 2)$ the curve is concave upward.

For $x = 1$, the abscissa of the maximum point, the second derivative is negative. This also shows that the curve is concave downward at the maximum point. Similarly, the substitution of 3 for x in (2) shows that the curve is concave upward at the minimum point, since this value of x renders the second derivative positive.

From this illustration and from the exercises of the preceding section it is obvious that, if the first derivative of a function is zero for a particular value of x, and if the second derivative is negative for the same value of x, this particular value of x corresponds to a maximum value of the function. Similarly, if for a particular value of x, the first derivative is zero and the second derivative is positive, this value of x corresponds to a minimum value of the function.

The second derivative test for maximum and minimum values of a function applies only if the second derivative has a value different from zero at a point where the first derivative vanishes.

If, for a particular value of x, both first and second derivatives become zero, the point on the curve corresponding to this value of x may be a maximum point, a minimum point, or a point of inflection. Consider, for example, the behavior of the functions $y = x^3$ and $y = x^4$ in the vicinity of $x = 0$.

Exercises

Sketch the following curves. Find their maximum and minimum points, and their points of inflection.

1. $y = x^3 - 3x + 1.$

2. $y = x^3 - 3x^2 + 4.$

3. $y = x^3 + 3x - 1.$

4. $y = x^3 + 3x^2 + 6x + 3.$

5. $3y = x^4 - 6x^2 + 8x + 12.$

6. $y = \dfrac{x^4}{12} - \dfrac{x^3}{3} + \dfrac{x^2}{2} - 3x + \frac{15}{4}.$

7. $y = \dfrac{8x}{x^2 + 4}.$

8. $2y = \dfrac{x^2}{x - 2}.$

49. Acceleration.[1] In §37 acceleration was defined as the time rate of change of velocity, *i.e.*, as the derivative of the velocity with respect to the time. But velocity is the derivative of distance with respect to time. Hence the acceleration is the second derivative of the distance with respect to the time. If s denotes the distance and t the time, the acceleration is expressed by $\frac{d^2s}{dt^2}$.

In the case of a freely falling body

$$\frac{d^2s}{dt^2} = g. \tag{1}$$

The relation between s and t can be found from this differential equation by performing two integrations, as follows:

The first integration gives

$$\frac{ds}{dt} = gt + C_1, \tag{2}$$

and the second

$$s = \tfrac{1}{2}gt^2 + C_1t + C_2. \tag{3}$$

Two arbitrary constants of integration are introduced. They can be determined by two conditions. If

$$s = s_0 \tag{4}$$

and

$$v = \frac{ds}{dt} = v_0 \tag{5}$$

when $t = 0$, (2) gives $C_1 = v_0$, and (3) gives $C_2 = s_0$. Then

$$v = gt + v_0 \tag{6}$$

and

$$s = \tfrac{1}{2}gt^2 + v_0t + s_0. \tag{7}$$

In §37, essentially the same method was used where the symbol $\frac{dv}{dt}$ was used for acceleration.

[1] The statements in this section refer to motion in a straight line.

Equations (6) and (7) give the velocity and the displacement each as a function of time. If t be eliminated between these equations, there results the relation

$$v^2 = 2gs + v_0^2 - 2gs_0, \tag{8}$$

an equation expressing velocity as a function of displacement.

Equation (8) can be obtained directly from equation (1) as follows:

Multiply each member of (1) by $2\frac{ds}{dt}$ and obtain

$$2\frac{ds}{dt}\frac{d^2s}{dt^2} = 2g\frac{ds}{dt}. \tag{9}$$

The first and second members of (9) are, respectively, the derivative of $\left(\frac{ds}{dt}\right)^2$ and $2gs$ with respect to t. We then have from (9)

$$\left(\frac{ds}{dt}\right)^2 = 2gs + C,$$

or

$$v^2 = 2gs + C. \tag{10}$$

If $v = v_0$ when $s = s_0$, $C = v_0^2 - 2gs_0$, and (10) becomes

$$v^2 = 2gs + v_0^2 - 2gs_0, \tag{11}$$

an equation identical with equation (8).

It is readily seen that the foregoing method can be used to find the velocity of any body whose acceleration is a known function of its displacement.

Illustration. Find the velocity of a body if it is known that its acceleration is proportional to its displacement s and directed in the opposite sense, and if $v = 2$ when $s = 1$.

The differential equation of motion is

$$\frac{d^2s}{dt^2} = -ks. \tag{12}$$

Multiply each member by $2\frac{ds}{dt}$ and obtain

$$2\frac{ds}{dt}\frac{d^2s}{dt^2} = -2ks\frac{ds}{dt},$$

which upon integration gives

$$v^2 = \left(\frac{ds}{dt}\right)^2 = C - ks^2.$$

Since $v = 2$ when $s = 1$,

$$C = k + 4$$

and

$$v^2 = k + 4 - ks^2. \tag{13}$$

Exercises

1. A body is thrown vertically upward with a velocity of 100 feet per second. Find its position as a function of the time.

2. A body is thrown vertically downward with a velocity of 100 feet per second from a captive balloon 2000 feet high. Find its position as a function of the time.

3. The acceleration of a body moving in a straight line is equal to $5t$. If $s = 0$ and $v = -10$ when $t = 0$, find the position of the body as a function of the time.

In each of the following exercises, find expressions for the square of the velocity, and determine the constant of integration under the conditions stated. The acceleration is represented by a.

4. $a = -k^2s^2$, and $v = 2$ when $s = 0$.

5. $a = k^2s$, and $v = 2$ when $s = 2$.

6. $a = \frac{k^2}{s^3}$, and $v = 3$ when $s = 1$.

7. $a = \frac{k^2}{s^2}$, and $v = 1$ when $s = 3$

8. $a = 3s^2 - 2s + 3$, and $v = 0$ when $s = 10$.

50. The Path of a Projectile. An interesting application of integration is found in the derivation of the equation of the path of a projectile, for instance, a baseball thrown with a given initial velocity having a given inclination to the horizontal.

Let O, Fig. 28, be the point from which the ball is thrown. Take this point as the origin of a system of rectangular coordinates with the X-axis horizontal and the Y-axis vertical. Let the ball be thrown so that, at the instant it leaves the hand, its direction of motion makes an angle α with the horizontal, and let the initial velocity of the ball be v_0. Choose as the origin of time the instant the ball is thrown. At this instant, $t = 0$, the x and y components of velocity are, respectively, $v_0 \cos \alpha$ and $v_0 \sin \alpha$. Let us now find the equation of the path of the ball assuming that the resistance of the air is negligible.

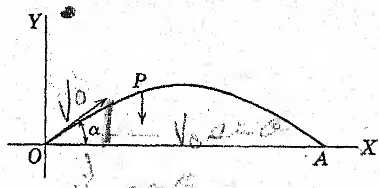

FIG. 28.

After the ball leaves the hand there is but one force acting upon it, the force of gravity acting downward. Consequently, the component of acceleration of the ball parallel to the X-axis is zero, while the component parallel to the Y-axis is $-g$. See §37. But the components of the acceleration of the ball parallel to the X- and Y-axes are $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$, respectively, where x and y are the coordinates of the position of the ball at any instant. Consequently

$$\frac{d^2x}{dt^2} = 0 \tag{1}$$

and

$$\frac{d^2y}{dt^2} = -g. \tag{2}$$

The integration of these equations, subject to the initial conditions, will give the position of the ball at any instant. Thus

$$\frac{dx}{dt} = C_1 \tag{3}$$

and

$$\frac{dy}{dt} = -gt + C_2. \tag{4}$$

The derivatives in (3) and (4) are, respectively, the x and y components of the velocity of the ball. Since these components are, respectively, $v_0 \cos \alpha$ and $v_0 \sin \alpha$ when $t = 0$, we have $C_1 = v_0 \cos \alpha$ and $C_2 = v_0 \sin \alpha$. Equations (3) and (4) become

$$\frac{dx}{dt} = v_0 \cos \alpha, \tag{5}$$

$$\frac{dy}{dt} = -gt + v_0 \sin \alpha. \tag{6}$$

By integrating (5) and (6) we obtain

$$x = (v_0 \cos \alpha)t + C_3 \tag{7}$$

and

$$y = -\tfrac{1}{2}gt^2 + (v_0 \sin \alpha)t + C_4. \tag{8}$$

Since $x = 0$ and $y = 0$ when $t = 0$, $C_3 = C_4 = 0$. Equations (7) and (8) become

$$x = (v_0 \cos \alpha)t, \tag{9}$$
$$y = -\tfrac{1}{2}gt^2 + (v_0 \sin \alpha)t, \tag{10}$$

the parametric equations of the path of the ball in terms of t.

By eliminating t between equations (9) and (10), we obtain

$$y = -\frac{g}{2v_0^2 \cos^2 \alpha} x^2 + x \tan \alpha, \tag{11}$$

the rectangular equation of the path. It is the equation of a parabola having its vertex at the point $\left(\frac{v_0^2 \sin 2\alpha}{2g}, \frac{v_0^2 \sin^2 \alpha}{2g}\right)$.

It is to be remembered that in the solution of this problem the resistance of the air was neglected.

Exercises

1. Show that $\frac{v_0^2 \sin^2 \alpha}{2g}$ is the height to which a ball will rise if thrown vertically upward with a velocity of $v_0 \sin \alpha$.

2. Find the angle of elevation, α, at which the ball must be thrown to make the range OA, Fig. 28, a maximum.

3. Find the equation of the locus of the highest point of the path of the ball, considering α the parameter. Discuss the locus.

4. A bomb is dropped from an aeroplane 3000 feet high and flying horizontally at a rate of 100 miles per hour. What angle should the line from the aeroplane to the target make with the vertical when the bomb is released? Neglect the resistance of the air.

5. The water in a tank resting on the ground is kept at a constant depth of 10 feet. A hole is bored in the side of the tank at a depth h below the surface of the water, h being so chosen that the water will strike the ground at the greatest distance from the tank. Find h if the velocity of the water as it flows out through the hole is equal to $\sqrt{2gh}$.

6. Water is discharged from a 4-inch horizontal pipe running full. The jet of water strikes the ground 10 feet beyond the end of the pipe. Find the discharge in cubic feet per second if the pipe is 8 feet above the ground.

7. During batting practice a ball is hit at a distance of 3 feet above the ground. Its initial velocity makes an angle of 45° with the horizontal. The ball just clears a 13-foot fence 410 feet away. Find the velocity with which the ball was batted.

8. A pitcher throws a ball horizontally 5 feet above the ground. The ball is caught 50 feet away one foot above the ground. Find the velocity with which the ball was thrown.

CHAPTER VI

APPLICATIONS

51. Area under a Curve: Rectangular Coordinates. An important application of the antiderivative is that of finding the area under a plane curve.

Let $APQB$, Fig. 29, be a continuous curve between the ordinates $x = a$ and $x = b$. Our problem is to find the area bounded by the curve, the X-axis, and the ordinates $x = a$ and $x = b$.

The area can be thought of as generated by a moving ordinate DP starting from $x = a$ and moving to the right. This ordinate

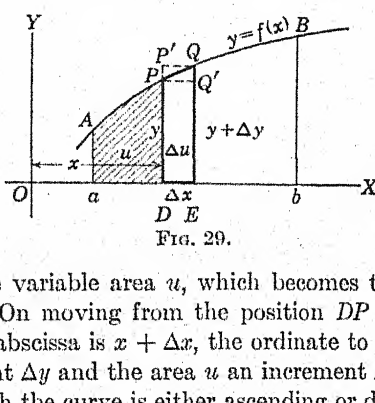

FIG. 29.

sweeps out the variable area u, which becomes the desired area when $x = b$. On moving from the position DP to the position EQ where the abscissa is $x + \Delta x$, the ordinate to the curve takes on an increment Δy and the area u an increment Δu. By taking Δx small enough the curve is either ascending or descending at all points between P and Q. It follows at once from the figure that

$$y\Delta x < \Delta u < (y + \Delta y)\Delta x, \tag{1}$$

or

$$y < \frac{\Delta u}{\Delta x} < y + \Delta y.$$

If the curve descends between P and Q, the signs of inequality in (1) are reversed. The argument which follows will not be affected.

As Δx approaches zero, Δy approaches zero and $y + \Delta y$ approaches y. Hence $\frac{\Delta u}{\Delta x}$, which lies between y and $y + \Delta y$, approaches y. Thus

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = y,$$

or

$$\frac{du}{dx} = y. = f(x) \tag{2}$$

If the equation of the curve is $y = f(x)$,

$$\mathbf{\frac{du}{dx} = f(x).} \tag{3}$$

Let $F(x)$ be a function whose derivative is $f(x)$. Then

$$u = F(x) + C.$$

The constant of integration is determined by the condition $u = 0$ when $x = a$. Hence

$$C = -F(a)$$

and

$$u = F(x) - F(a), \tag{4}$$

an expression for the variable area measured from the ordinate $x = a$ to the variable ordinate whose abscissa is x. The area sought is obtained by putting $x = b$ in equation (4).

$$\mathbf{u = F(b) - F(a).} \tag{5}$$

If the curve lies entirely below the X-axis between $x = a$ and $x = b$, $\frac{du}{dx} = y$ will be negative and the area as computed by (5)

will be negative. Obviously, then, if the curve lies partly above and partly below the X-axis, the area as computed by (5) will be the algebraic sum of the areas lying above and below the X-axis, those above being reckoned positive, those below negative.

Illustration. Find the area bounded by $y = x^2$, the X-axis and the ordinates $x = 2$ and $x = 4$.

$$\frac{du}{dx} = x^2.$$

$$u = \tfrac{1}{3}x^3 + C.$$

When $x = 2$, $u = 0$, and $C = -\tfrac{8}{3}$. Then

$$u = \tfrac{1}{3}x^3 - \tfrac{8}{3}.$$

When $x = 4$,

$$u = \tfrac{64}{3} - \tfrac{8}{3} = \tfrac{56}{3} = 18.67.$$

Exercises

In each of the following four exercises, find the area bounded by the curve, the X-axis, the line $x = 2$, and the line $x = 4$:

1. $y = 2x^2$.

2. $y = x^2 - 6x + 5$.

3. $y = x^2 + 4x + 1$.

4. $y = x^2 - 6x + 8$.

5. Find the area bounded by the curve $y = 1 - x^2$ and the X-axis.

6. Find the area bounded by the curve $y = x^2 - 9$ and the X-axis.

7. Find the area bounded by the curve $x = 4 - y^2$ and the Y-axis.

HINT. Integrate with respect to y, *i.e.*, use $\dfrac{du}{dy} = f(y)$.

8. Find the area bounded by the curve $x = y^2 - 25$ and the Y-axis.

In each of Exercises 9 to 17, find the area completely bounded by the two given curves.

9. $y^2 = 4x$ and $4y = x^2$.

10. $y^2 - 2y - 4x + 9 = 0$ and $x^2 - 4x - 4y + 8 = 0$.

11. $y^2 = 5x^3$ and $y = 5x$.

12. $(y - 1)^2 = 5(x - 1)^3$ and $y = 5x - 4$.

13. $y^2 = 16x$ and $y + 4x = 3$.

HINT. Integrate with respect to y.

14. $(y - 2)^2 = 9(x - 2)$ and $y + 2x = 6$.

15. $y^2 = 1 + x$ and $y^2 = 3 - x$.

16. $y^2 = 3x$ and $y^2 = 4(x - 3)$.

17. $y^2 = 2x$ and $y^2 = 4(x - 4)$.

52. Work Done by a Variable Force. In this section there is given a method of finding the work done by a variable force whose line of action remains unchanged.

Illustrations of such variable forces are:

1. The force necessary to stretch a bar from its original length l to a length $l + s$, which is a function of the elongation s.

Let AB, Fig. 30, represent a bar of length l, held fast at the left-hand end A. If a force f acting to the right is applied at the right-

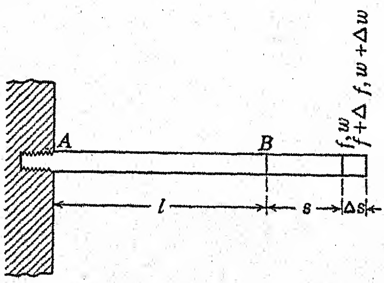

FIG. 30.

hand end B, the bar will stretch to a length represented by $l + s$ in the figure. It is shown experimentally that up to a certain limit, the elongation is proportional to the force applied (Hooke's law), *i.e.*,

$$f = ks,$$

where k is a constant depending upon the length, the cross section, and the material of the bar.

2. The force exerted on the piston of a steam engine by expanding steam, which is, after cutoff, a function of the distance of the piston from one end of the cylinder.

3. The force of attraction between two distant masses, m and M, which is given by Newton's law

$$f(s) = \frac{kmM}{s^2},$$

where s is the distance between the masses and k is a factor of proportionality. Note that the equation is of the form

$$f(s) = \frac{a}{s^2}.$$

The work done by a constant force in producing a certain displacement of its point of application in its line of action is, by definition, the product of the force by the displacement. If, however, the force is not constant but variable, as in each of the three cases cited above, the work done by the force cannot be found by the simple process of multiplication. A process will now be developed for finding the work done by a variable force. Three illustrative examples, one for each of the three typical forces discussed at the beginning of the section, will be worked out in detail. After this is done the method of finding the work done by any variable force should be obvious.

Illustration 1. Find the work done in stretching a spring, whose original (unstretched) length is 100 inches, from a length of 101 inches to a length of 104 inches, if it is known that a force of 50 pounds will stretch the spring to a length of 101 inches.

Let w represent the work done by the variable force $f = ks$ in stretching the bar from a length l to a length $l + s$. Let Δw denote the work done in producing the additional elongation Δs, and let $f + \Delta f$ be the force necessary to maintain the elongation $s + \Delta s$. A figure similar to Fig. 30 may be used.

In producing the elongation Δs the force varied from f to $f + \Delta s$, and hence the work Δw lies between $f\Delta s$ and $(f + \Delta f)\Delta s$, expressions for the work which would have been done had the forces f and $f + \Delta s$, respectively, acted through the distance Δs.

Then

$$f\Delta s < \Delta w < (f + \Delta f)\Delta s,$$

or

$$f < \frac{\Delta w}{\Delta s} < f + \Delta f.$$

As Δs approaches zero, Δf approaches zero, and $\frac{\Delta w}{\Delta s}$ approaches f. Hence

$$\frac{dw}{ds} = f = ks.$$

Integration gives

$$w = \frac{ks^2}{2} + C. \tag{1}$$

Since $w = 0$ when $s = 1$, $C = -\frac{k}{2}$ and equation (1) becomes

$$w = \frac{k}{2}(s^2 - 1). \tag{2}$$

Since $f = 50$ when $s = 1$, it follows from the relation $f = ks$, that $k = 50$. On replacing k by this value, equation (2) becomes

$$w = 25(s^2 - 1),$$

an expression representing the work done in producing an elongation s. Since our problem was to find the work done in producing an elongation of 4 inches, we substitute 4 for s and obtain

$$w = 375.$$

The work performed is then 375 inch-pounds.

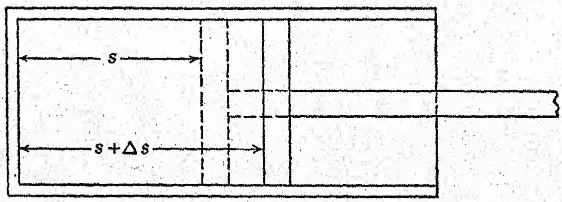

FIG. 31.

Illustration 2. Gas is enclosed in a cylinder, one end of which is closed by a movable piston. Find the work done by the gas in

expanding in accordance with the law $pv^{1.4} = K$, from a volume of 3 cubic feet at a pressure of 15,000 pounds per square foot to a volume of 4 cubic feet.

Let A be the area of the cross section of the cylinder, Fig. 31. The force acting on the piston is pA. Noting that Δp is negative when Δs is positive,

$$(p + \Delta p)A\Delta s < \Delta w < pA\Delta s.$$

Since $A\Delta s$ is equal to the increase, Δv, in the volume v, this inequality can be written

$$(p + \Delta p)\Delta v < \Delta w < p\Delta v$$

or

$$p + \Delta p < \frac{\Delta w}{\Delta v} < p.$$

Hence

$$\frac{dw}{dv} = p,$$

or

$$\frac{dw}{dv} = \frac{K}{v^{1.4}}.$$

Integrating,

$$w = -\frac{1}{0.4}\frac{K}{v^{0.4}} + C.$$

When $v = 3$, $w = 0$. Hence

$$C = \frac{1}{0.4}\frac{K}{3^{0.4}},$$

and

$$w = \frac{K}{0.4}\left[\frac{1}{3^{0.4}} - \frac{1}{v^{0.4}}\right].$$

When $v = 4$,

$$w = \frac{K}{0.4}\left[\frac{1}{3^{0.4}} - \frac{1}{4^{0.4}}\right].$$

When $v = 3$, $p = 15{,}000$. Hence

$$K = (15{,}000)(3^{1.4})$$

and

$$\begin{aligned} w &= \frac{15{,}000}{0.4}3^{1.4}\left[\frac{1}{3^{0.4}} - \frac{1}{4^{0.4}}\right] \\ &= \frac{450{,}000}{4}\left[1 - (0.75)^{0.4}\right] \\ &= 12{,}230. \end{aligned}$$

Illustration 3. Two masses M and m are supposed concentrated at the points A and B, respectively. Find the work done against the force of attraction in moving the mass m along the line AB from a distance a to a distance b from the mass M, the latter mass being fixed.

If f is the force of attraction between the two masses,

$$f = \frac{kmM}{s^2}.$$

The student will set up inequalities similar to those of *Illustration* 1 and show that

$$\frac{dw}{ds} = \frac{kmM}{s^2}. \tag{3}$$

Integrating,

$$w = -\frac{kmM}{s} + C.$$

Since $w = 0$ when $s = a$, $C = \dfrac{kmM}{a}$, and

$$w = kmM\left[\frac{1}{a} - \frac{1}{s}\right]. \tag{4}$$

To find the work done in moving the mass m from A to B, substitute b for s in (4). Then

$$w = kmM\left[\frac{1}{a} - \frac{1}{b}\right]. \tag{5}$$

If $m = 1$ and if b increases without limit, equation (5) becomes

$$w = \frac{kM}{a}.$$

This magnitude is called the potential at the point A of the field of force due to the mass M.

Exercises

1. Find the work done in stretching a spring from a length of 24 inches to a length of 28 inches, if the length of the spring is 20 inches when no force is applied and if a force of 20 pounds is necessary to stretch it from a length of 20 inches to a length of 22 inches.

2. Find the work done in compressing a spring 5 inches long to a length of 4.5 inches if a force of 2500 pounds is necessary to compress it to a length of 4 inches.

3. A force of 100 pounds will stretch a spring 24 inches long to a length of 25 inches. Find the work done in stretching this spring from a length of 25 inches to a length of 28 inches.

4. In the case of a bar under tension, Fig. 30, the relation between the stretching force f, the original length of the bar l, and the elongation of the bar s is given by

$$f = \frac{EAs}{l},$$

where E is the modulus of elasticity of the material of the bar and A is the area of the cross section of the bar. Find the work done in stretching a round iron rod $\frac{1}{2}$ inch in diameter and 5 feet long to a length of 60.5 inches, given that $E = 3 \cdot 10^7$ pounds per square inch.

5. A spherical conductor A is charged with positive electricity and a second spherical conductor B with negative electricity. The force of attraction between them varies inversely as the square of the distance between their centers. If the force is 25 dynes when the centers are 40 centimeters apart, find the work done by the force of attraction in changing the distance between the centers from 50 to 30 centimeters.

6. Find the work done by a gas in expanding in accordance with the law $pv^{\frac{4}{3}} = C$ from a volume of 4 cubic feet to one of 6 cubic feet, if $p = 80$ pounds per square inch when $v = 5$ cubic feet.

7. Find the original length x of a spring, if 15 and 32 inch-pounds of work are expended, respectively, in stretching it from a length of

11 inches to a length of 14 inches, and from a length of 12 inches to a length of 16 inches.

8. Find the work done in stretching the spring of Exercise 7 from a length of 13 inches to a length of 15 inches.

9. Find the work done by a gas, expanding in accordance with the law $pv^{1.4} = K$, from a volume of 5 cubic feet to one of 7 cubic feet, if the pressure p is 50 pounds per square inch when the volume is 6 cubic feet.

10. Find the original length x of a spring, if 75 and 72 inch-pounds of work are expended, respectively, in stretching it from a length of 20 inches to a length of 25 inches, and from a length of 22 inches to a length of 26 inches.

53. Parabolic Cable. Suppose a cable AOB, Fig. 32, loaded uniformly along the horizontal, *i.e.*, any segment of the cable sustains a weight proportional to the projection of the segment upon a horizontal line. Let k be the weight carried by a portion of the cable whose horizontal projection is one unit of length.

Choose O, the lowest point of the cable, as origin and a horizontal line through O as axis of x. Let P be any point on the cable.

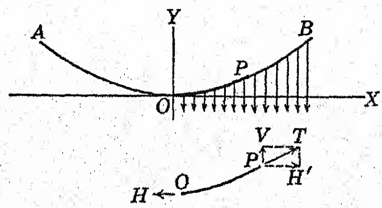

FIG. 32.

Suppose the portion OP of the cable cut free, Fig. 32. To keep this portion in a state of equilibrium a horizontal force H and an inclined force T must be introduced at the points O and P, respectively. The force H must be equal in magnitude to the tension in the cable at O, and it must act in the direction of the tangent line at that point. Similarly, the force T must be equal to the tension in the cable at the point P and act in the direction of the tangent line. The force T can be resolved into its vertical and horizontal components V and H', respectively. Now H and H' are the only horizontal components of the forces acting on OP and, since

OP is in a state of equilibrium, they must balance each other. Therefore,

$$H = H'. \tag{1}$$

Hence the horizontal component of the tension in the cable is independent of the point P, *i.e.*, it is a constant.

In like manner the only vertical components of the forces acting on OP are the weight kx supported by OP, acting downward, and V, the vertical component of T. They must balance one another. Hence

$$V = kx. \tag{2}$$

The slope of the tangent line to the curve at the point P is $\frac{V}{H'}$. Then

$$\frac{dy}{dx} = \frac{V}{H'} = \frac{kx}{H}. \tag{3}$$

This is the slope of the curve at any point. On integrating we obtain the equation of the curve apart from the arbitrary constant C.

$$y = \frac{kx^2}{2H} + C. \tag{4}$$

C is determined by the condition that $y = 0$ when $x = 0$. Then $C = 0$, and (4) becomes

$$y = \frac{kx^2}{2H}. \tag{5}$$

This is the equation of a parabola with its vertex at the origin.

CHAPTER VII

INFINITESIMALS, DIFFERENTIALS, DEFINITE INTEGRALS

54. Infinitesimals. *In* §24 *an infinitesimal was defined as a variable whose limit is zero.* Thus, $x^2 - 1$, as x approaches 1, is an infinitesimal.

It is to be noted that a variable is thought of as an infinitesimal only when it is in the state of approaching zero. Thus $x^2 - 1$ is an infinitesimal only when x approaches $+1$ or -1. An infinitesimal has two characteristic properties: (1) It is a variable. (2) It approaches the limit zero; *i.e.*, the conditions of the problem are such that the numerical value of the variable can be made less than any preassigned positive number, however small.

This meaning of the word "infinitesimal" in mathematics is entirely different from its meaning in everyday speech. When we say in ordinary language that a quantity is infinitesimal, we mean that it is very small. But it is a constant magnitude and not one whose numerical measure can be made less than any preassigned positive number, however small. Thus, 0.000001 of a milligram of salt might be spoken of as an infinitesimal quantity of salt, but the number 0.000001 is clearly not an infinitesimal in the sense of the mathematical definition. On the other hand, if we have a solution containing a certain amount of salt per cubic centimeter and allow pure water to flow into the vessel containing the solution while the solution flows off through an overflow pipe, the quantity of salt per cubic centimeter constantly diminishes. The amount of salt left in solution in the vessel after a time t is then an infinitesimal, as t becomes infinite.

Infinitesimals are of fundamental importance in the calculus. The derivative, which we have already used in studying functions, is the limit of the ratio of two infinitesimals, Δy and Δx.

55. $\lim\limits_{\alpha \to 0} \dfrac{\sin \alpha}{\alpha}$. Let the arc AB, Fig. 33, subtend an angle α at the center O of a circle of radius r. The angle α is measured in radians. Let AT be tangent to the circle at A and let BC be perpendicular to OA. The area of the triangle OCB is less than the area of the circular sector OAB, and this, in turn, is less than the area of the triangle OAT.

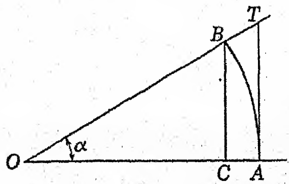

FIG. 33.

$$\tfrac{1}{2}(BC)(OC) < \tfrac{1}{2}\alpha r^2 < \tfrac{1}{2}(AT)r$$

$$\frac{BC}{r}\frac{OC}{r} < \alpha < \frac{AT}{r}.$$

$$\sin\alpha\cos\alpha < \alpha < \tan\alpha \tag{1}$$

$$\cos\alpha < \frac{\alpha}{\sin\alpha} < \frac{1}{\cos\alpha}. \tag{2}$$

As the angle α approaches zero as a limit, $\cos\alpha$ approaches the limit 1. Hence the first and last members of the inequalities (2) approach the same limit, 1. Then the second member, $\dfrac{\alpha}{\sin\alpha}$, which lies between them, must approach the limit, 1. Therefore

$$\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha} = \lim_{\alpha\to 0}\frac{1}{\dfrac{\alpha}{\sin\alpha}} = 1. \tag{3}$$

It is to be noted that in obtaining this limit the angle α was taken to be measured in radians. If α were measured in degrees the limit would be $\dfrac{\pi}{180}$ instead of 1.

56. $\lim\limits_{\alpha\to 0}\dfrac{\tan\alpha}{\alpha}$, $\lim\limits_{\alpha\to 0}\dfrac{\tan\alpha}{\sin\alpha}$.

$$\begin{aligned}\lim_{\alpha\to 0}\frac{\tan\alpha}{\alpha} &= \lim_{\alpha\to 0}\left(\frac{\sin\alpha}{\alpha}\,\frac{1}{\cos\alpha}\right)\\ &= \left(\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}\right)\left(\lim_{\alpha\to 0}\frac{1}{\cos\alpha}\right)\\ &= 1.\end{aligned} \tag{1}$$

$$\lim_{\alpha\to 0}\frac{\tan\alpha}{\sin\alpha}=\lim_{\alpha\to 0}\frac{1}{\cos\alpha}=1. \tag{2}$$

57. $\lim\limits_{\alpha\to 0}\dfrac{1-\cos\alpha}{\alpha}$.

Since

$$\frac{1-\cos\alpha}{\alpha}=\frac{2\sin^2\frac{\alpha}{2}}{\alpha}=\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}\sin\frac{\alpha}{2},$$

$$\lim_{\alpha\to 0}\frac{1-\cos\alpha}{\alpha}=\left(\lim_{\alpha\to 0}\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}\right)\left(\lim_{\alpha\to 0}\sin\frac{\alpha}{2}\right).$$

The limits of the first and second factors are 1 and 0, respectively (see §55).
Hence

$$\lim_{\alpha\to 0}\frac{1-\cos\alpha}{\alpha}=0. \tag{1}$$

In Fig. 33, AB, AC, AT, BT, and BC are infinitesimals as α approaches zero. Then,

$$\text{from (3), §55, } \lim_{\alpha\to 0}\frac{BC}{AB}=1,$$

$$\text{from (1), §56, } \lim_{\alpha\to 0}\frac{AT}{AB}=1,$$

$$\text{from (2), §56, } \lim_{\alpha\to 0}\frac{AT}{BC}=1,$$

$$\text{from (1), §57, } \lim_{\alpha\to 0}\frac{AC}{AB}=0.$$

58. Order of Infinitesimals. Consider the infinitesimals x^2 and x as x approaches zero. The ratio of x^2 to x is x, which is itself an infinitesimal. The infinitesimals x^2 and x are repre-

sented, Fig. 34, by the ordinates MP and MN, to the curves $y = x^2$ and $y = x$. The quotient

$$\frac{x^2}{x} = \frac{MP}{MN}$$

is a measure of the relative magnitude of these infinitesimals as they approach zero.

On the other hand, the infinitesimals $2x$ and x behave very differently. Their quotient is $\frac{2x}{x} = 2$, and the limit of this

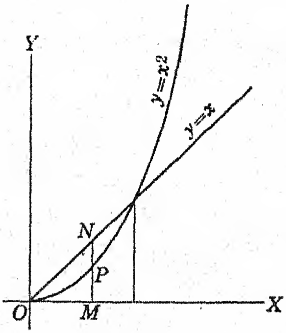

FIG. 34.

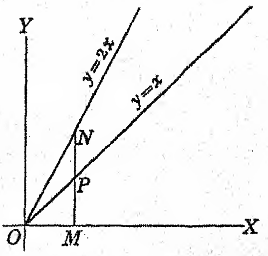

FIG. 35.

quotient is 2. In this case the limit of the ratio of the infinitesimals is not zero (see Fig. 35).

Again,

$$\lim_{\alpha \to 0} \frac{1 - \cos \alpha}{\alpha} = 0,$$

while

$$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1.$$

These illustrations of the comparison of two infinitesimals lead to the following definitions of the order of one infinitesimal with respect to another.

Two infinitesimals, α and β, are said to be of the same order if the limit of $\frac{\alpha}{\beta}$ is a finite number not zero.

If the limit of $\frac{\alpha}{\beta}$ is zero, α is said to be of higher order than β.

Thus, $2x$ and x are of the same order; x^2 is an infinitesimal of higher order than x; $\sin \alpha$ and α, or CB and AB, Fig. 33, are of the same order; $\tan \alpha$ and α, or AT and AB, Fig. 33, are of the same order; $\tan \alpha$ and $\sin \alpha$, or AT and CB, Fig. 33, are of the same order; $1 - \cos \alpha$ is of higher order than α, or CA, Fig. 33, is of higher order than AB.

If lim $\frac{\alpha}{\beta} = 1$, β and α differ by an infinitesimal of higher order, for

$$\lim \frac{\beta - \alpha}{\alpha} = \lim \frac{\beta}{\alpha} - 1 = 1 - 1 = 0.$$

Hence $\beta - \alpha$ is of higher order than α and consequently also of higher order than β, which is of the same order as α.

Illustration 1. Show that $\sin \theta - \theta$ is an infinitesimal of higher order than θ as θ approaches zero.

$$\begin{aligned}\lim_{\theta \to 0} \frac{\sin \theta - \theta}{\theta} &= \lim_{\theta \to 0} \left[\frac{\sin \theta}{\theta} - 1\right] = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} - 1 \\ &= 1 - 1 = 0.\end{aligned}$$

Hence $\sin \theta - \theta$ is an infinitesimal of higher order than θ as θ approaches zero.

Illustration 2. Show that $1 - \cos \theta$ is an infinitesimal of higher order than $\cos \theta \sin \theta$ as θ approaches zero.

$$\begin{aligned}\lim_{\theta \to 0} \frac{1 - \cos \theta}{\cos \theta \sin \theta} &= \lim_{\theta \to 0} \frac{1 - \cos^2 \theta}{\cos \theta \sin \theta (1 + \cos \theta)} \\ &= \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta (1 + \cos \theta)} = 0.\end{aligned}$$

Hence $1 - \cos \theta$ is an infinitesimal of higher order than $\cos \theta \sin \theta$ as θ approaches zero.

Illustration 3. Show that $1 - \sin\theta$ and $\cos^2\theta$ are infinitesimals of the same order as θ approaches $\frac{\pi}{2}$.

$$\lim_{\theta\to\frac{\pi}{2}} \frac{1-\sin\theta}{\cos^2\theta} = \lim_{\theta\to\frac{\pi}{2}} \frac{1-\sin\theta}{1-\sin^2\theta} = \lim_{\theta\to\frac{\pi}{2}} \frac{1}{1+\sin\theta} = \tfrac{1}{2}.$$

Since the limit is a finite number, not zero, the infinitesimals are of the same order.

Illustration 4. Find $\lim\limits_{x\to 1} \dfrac{x^2-1}{x^2+x-2}$.

$$\lim_{x\to 1} \frac{x^2-1}{x^2+x-2} = \lim_{x\to 1} \frac{x+1}{x+2} = \tfrac{2}{3}.$$

Exercises

Find the value of each of the expressions given in Exercises 1 to 12.

1. $\lim\limits_{\theta\to 0} \dfrac{\sin\theta - \theta}{\sin\theta}$.
2. $\lim\limits_{\theta\to 0} \dfrac{\tan\theta - \theta}{\theta}$.
3. $\lim\limits_{\theta\to 0} \dfrac{\tan\theta + \theta}{\tan\theta}$.
4. $\lim\limits_{\theta\to 0} \dfrac{\tan\theta + \theta}{\sin\theta}$.
5. $\lim\limits_{\theta\to 0} \dfrac{\sin\theta + \tan\theta}{\tan\theta}$.
6. $\lim\limits_{\theta\to 0} \dfrac{\sin\theta - \tan\theta}{\sin\theta}$.
7. $\lim\limits_{\theta\to\frac{\pi}{2}} \dfrac{\sqrt{1-\sin\theta}}{\cot\theta}$.
8. $\lim\limits_{\alpha\to\frac{\pi}{2}} \dfrac{1-\cos\alpha-\sin\alpha}{\cos\alpha}$.
9. $\lim\limits_{\alpha\to\frac{\pi}{2}} \dfrac{\cos\alpha}{\alpha-\frac{\pi}{2}}$.
10. $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{1-\tan x}{\cos 2x}$.
11. $\lim\limits_{x\to 3} \dfrac{x^2-x-6}{x^2-9}$.
12. $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x^3-2x^2-x+2}$.

13. Show that $x - 2x^2$ and $3x + x^3$ are infinitesimals of the same order as x approaches zero.

14. Show that $1 - \sin\theta$ is an infinitesimal of higher order than $\cos\theta$ as θ approaches $\frac{\pi}{2}$.

15. Show that $\sec\alpha - \tan\alpha$ is an infinitesimal as α approaches $\frac{\pi}{2}$.

16. Show that $1 - \sin\alpha$ is an infinitesimal of higher order than $\sec\alpha - \tan\alpha$ as α approaches $\frac{\pi}{2}$.

17. Show that $1 - \cos\theta$ is an infinitesimal of the same order as θ^2 as θ approaches zero.

59. Theorem. *The limit of the quotient of two infinitesimals, α and β, is not altered if they are replaced by two other infinitesimals, γ and δ, respectively, such that* $\lim \frac{\alpha}{\gamma} = 1$ *and* $\lim \frac{\beta}{\delta} = 1.$

PROOF:

$$\frac{\alpha}{\beta} = \frac{\gamma \dfrac{\alpha}{\gamma}}{\delta \dfrac{\beta}{\delta}},$$

$$\lim \frac{\alpha}{\beta} = \frac{\lim \dfrac{\alpha}{\gamma}}{\lim \dfrac{\beta}{\delta}} \lim \frac{\gamma}{\delta} = \lim \frac{\gamma}{\delta},$$

since

$$\lim \frac{\alpha}{\gamma} = \lim \frac{\beta}{\delta} = 1.$$

It is evident from the proof that the limit of the quotient is unaltered if only one of the infinitesimals, say α, is replaced by another infinitesimal γ, such that $\lim \frac{\gamma}{\alpha} = 1.$

Illustrations.

1. Since

$$\lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = 1,$$

$$\lim_{\alpha \to 0} \frac{1 - \cos\alpha}{\sin\alpha} = \lim_{\alpha \to 0} \frac{1 - \cos\alpha}{\alpha} = 0.$$

2. Since

$$\lim_{\alpha \to 0} \frac{\tan\alpha}{\alpha} = 1,$$

$$\lim_{\alpha \to 0} \frac{1 - \cos\alpha}{\tan\alpha} = \lim_{\alpha \to 0} \frac{1 - \cos\alpha}{\alpha} = 0.$$

3. In Fig. 33,

$$\lim_{\alpha \to 0} \frac{CA}{AB} = \lim_{\alpha \to 0} \frac{CA}{BC} = \lim_{\alpha \to 0} \frac{CA}{AT} = 0.$$

Exercises

Find each of the following limits:

1. $\lim_{\alpha \to 0} \frac{\tan 5\alpha}{\sin 3\alpha}$.

2. $\lim_{\alpha \to 0} \frac{1 - \cos \alpha}{\sin \alpha}$.

3. $\lim_{\alpha \to 0} \frac{\sin 3\alpha}{\sin 5\alpha}$.

4. $\lim_{\alpha \to 0} \frac{\sin \alpha(1 - \cos \alpha)}{\alpha^3}$.

5. $\lim_{\alpha \to 0} \frac{(\alpha - 5)^2 \sin \alpha}{\tan \alpha}$.

6. $\lim_{\alpha \to 0} \frac{1 + \tan \alpha - \cos \alpha}{\sin \alpha}$.

7. $\lim_{\alpha \to 0} \frac{\sin^2 3\alpha}{\alpha^2}$.

8. $\lim_{\alpha \to 0} \frac{\alpha^3}{1 - \cos \alpha}$.

9. $\lim_{\alpha \to 0} \frac{(\alpha - 3)^2(1 - \cos \alpha)}{\alpha^2}$.

10. Show that $\lim_{x \to 0} \frac{3x^2 - 4x^3}{2x^2 - 5x^4} = \frac{3}{2}$.

HINT. Replace numerator by $3x^2$ and denominator by $2x^2$.

11. Show that $\lim_{x \to \infty} \frac{\frac{2}{x^2} + \frac{3}{x^4}}{\frac{1}{x^2} + \frac{4}{x^3}} = \lim_{x \to \infty} \frac{\frac{2}{x^2}}{\frac{1}{x^2}} = 2.$

60. Differentials. Let PT, Fig. 36, be a tangent line drawn to the curve $y = f(x)$ at the point P. Let $DE = \Delta x$, $RQ = \Delta y$, and let angle $RPT = \tau$.

From the figure,

$$\frac{RM}{\Delta x} = \tan \tau = f'(x),$$

or

$$RM = f'(x)\Delta x.$$

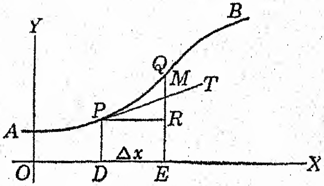

FIG. 36.

This is the increment which the function would take on if it were to change uniformly at a rate equal to its rate of change at P.

This quantity, $f'(x)\Delta x$, is called differential y, and is denoted by dy. Its defining equation is

$$dy = f'(x)\Delta x. \tag{1}$$

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Δx, *the increment of the independent variable*, is called differential x and is denoted by dx, *i.e.*, $\Delta x = dx$. Equation (1) becomes

$$dy = f'(x)dx^1. \tag{2}$$

In Fig. 36, $RM = dy$ and $DE = PR = dx$.

In general, dy is not equal to Δy, the difference being represented by the line MQ. However, it will be shown that this difference is an infinitesimal of higher order than Δy and dy.

From the relation

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x),$$

it follows that

$$\frac{\Delta y}{\Delta x} = f'(x) + \epsilon,$$

where

$$\lim_{\Delta x \to 0} \epsilon = 0.$$

Then

$$\begin{aligned} \Delta y &= f'(x)\Delta x + \epsilon\,\Delta x \\ &= dy + \epsilon\,\Delta x \end{aligned}$$

and

$$\Delta y - dy = \epsilon\,\Delta x.$$

This difference, $\epsilon\,\Delta x$, is an infinitesimal of higher order than Δx since

$$\lim_{\Delta x \to 0} \frac{\epsilon \Delta x}{\Delta x} = \lim_{\Delta x \to 0} \epsilon = 0.$$

The difference is also an infinitesimal of higher order than Δy or dy which are, in general, infinitesimals of the same order as Δx.

[1] In the expression (2) for the differential of the function $f(x)$, the first derivative is the coefficient of the differential of the argument, and for this reason it is sometimes called the *differential coefficient*.

To illustrate, consider the function $y = x^3$ (see Fig. 37).

$$dy = 3x^2\,dx$$
$$\Delta y = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3.$$

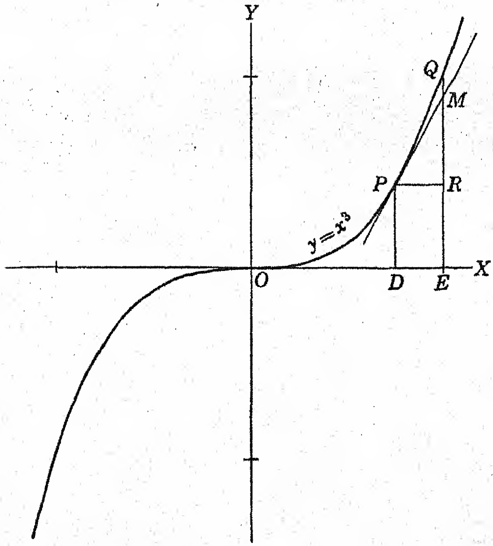

FIG. 37.

Since $dx = \Delta x$,

$$\Delta y - dy = 3x(\Delta x)^2 + (\Delta x)^3.$$

This difference is an infinitesimal of higher order than either Δy or dy.

Thus if $x = 2$ and $dx = \Delta x = 0.1$,

$$dy = (3)(4)(0.1) = 1.2,$$

and

$$\Delta y = (3)(4)(0.1) + (3)(2)(0.01) + 0.001 = 1.261.$$

dy differs from Δy by about 5 per cent of its value.

If $x = 2$ and $\Delta x = 0.01$,

$$dy = (3)(4)(0.01) = 0.12$$

and

$$\Delta y = (3)(4)(0.01) + (3)(2)(0.0001) + 0.000001$$
$$= 0.120601.$$

dy differs from Δy by about 0.5 of 1 per cent of its value. The last two terms in the expression for Δy have relatively less effect than in the former case.

It is to be noted that dx, the differential of the independent variable x, is an arbitrary increment of this variable, and that dy, the differential of the dependent variable y, is determined in accordance with the definition (2) as the product of the increment dx and the derivative $f'(x)$, *i.e.*, as the product of the increment dx and the slope of the tangent to the curve $y = f(x)$ at the point at which the differential is computed. Thus we can divide (2) by dx and obtain

$$\frac{dy}{dx} = f'(x), \tag{3}$$

where dy and dx denote the differentials of y and x, respectively. Thus from the definition of differentials the first derivative may be regarded as the quotient of the differential of y by the differential of x.

It is to be observed, however, that this statement gives no new meaning to the derivative, since the derivative was used in the definition of the differential.

In the case of the following functions find dy and Δy for the given values of x and dx. Tabulate your results.

1. $y = x^2$

$x = 1$, $dx = 0.5$; $x = 1$, $dx = 0.1$; $x = 1$, $dx = 0.01$;
$x = 5$, $dx = 0.5$; $x = 5$, $dx = 0.1$; $x = 5$, $dx = 0.01$;
$x = 10$, $dx = 2$; $x = 10$, $dx = 0.1$; $x = 10$, $dx = 0.01$.

2. $y = \sqrt{25 - x^2}$

$x = 3, dx = 1; x = 3, dx = 0.2; x = 3, dx = 0.1; x = 3, dx = 0.01;$
$x = 4, dx = 1; x = 4, dx = 0.2; x = 4, dx = 0.1; x = 4, dx = 0.01.$

61. Formulas for the Differentials of Functions. In accordance with equation (3) of the preceding section, any formula involving first derivatives may be regarded as a formula in which each first derivative is replaced by the quotient of the corresponding differentials. Thus,

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Each derivative being considered as a fraction whose denominator is dx, we can multiply by dx, and obtain

$$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}.$$

In words, *the differential of a fraction is equal to the denominator times the differential of the numerator minus the numerator times the differential of the denominator, all divided by the square of the denominator.* It will be noted that the wording is the same as that for the derivative of a fraction except that throughout the word *differential* replaces the word *derivative.*

The other formulas for derivatives which have been developed are expressed below with the corresponding formulas for differentials.

Formulas

1. $\frac{dc}{dx} = 0.$ $\qquad dc = 0.$

2. $\frac{d(cu)}{dx} = c\frac{du}{dx}.$ $\qquad d(cu) = c\,du.$

3. $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$ $\qquad d(u+v) = du + dv.$

4. $\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}.$ $\qquad du^n = nu^{n-1}\,du.$

5. $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$ $\qquad d(uv) = u\,dv + v\,du.$

6. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$ $\qquad d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$

7. $\dfrac{d\left(\dfrac{c}{v}\right)}{dx} = -\dfrac{c\dfrac{dv}{dx}}{v^2}$. $\qquad d\left(\dfrac{c}{v}\right) = -\dfrac{c\,dv}{v^2}$.

8. $\dfrac{d\left(\dfrac{c}{v^n}\right)}{dx} = -\dfrac{cn\dfrac{dv}{dx}}{v^{n+1}}$. $\qquad d\left(\dfrac{c}{v^n}\right) = -\dfrac{cn\,dv}{v^{n+1}}$.

9. $\dfrac{du^{\frac{1}{2}}}{dx} = \dfrac{\dfrac{du}{dx}}{2u^{\frac{1}{2}}}$ $\qquad du^{\frac{1}{2}} = \dfrac{du}{2u^{\frac{1}{2}}}$.

The formula for the differential of $y = cu^n$ can be put in the following convenient form:

10. $\dfrac{dy}{y} = n\dfrac{du}{u}$,

which is obtained directly by dividing $dy = cnu^{n-1}\,du$ by $y = cu^n$.

The process of finding either the derivative or the differential of a function is called differentiation.

The process of finding a function when its derivative or its differential is given is called integration.

We have no symbol representing integration when applied to derivatives. The symbol for integration when applied to differentials is $\int$. Thus $\int 3x^2\,dx = x^3 + C$.

The left-hand member is read "the integral of $3x^2\,dx$."

Illustrations.

1. If $y = \sqrt{1 - x^2}$,

$$dy = \tfrac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x\,dx)$$
$$= -\frac{x\,dx}{\sqrt{1 - x^2}}.$$

By formula 10, where $\quad u = 1 - x^2$,

$$\frac{dy}{y} = \frac{1}{2}\,\frac{-2xdx}{(1 - x^2)}$$
$$= -\frac{x\,dx}{1 - x^2}.$$

2. If $y = \dfrac{x}{x^2 - 1}$,

$$\begin{aligned} dy &= \frac{(x^2 - 1)dx - xd(x^2 - 1)}{(x^2 - 1)^2} \\ &= \frac{(x^2 - 1)dx - x(2x\,dx)}{(x^2 - 1)^2} \\ &= -\frac{(x^2 + 1)dx}{(x^2 - 1)^2}. \end{aligned}$$

3. If $x^2 + xy^2 - 3x + 2y - 1 = 0$

$$\begin{aligned} 2x\,dx + 2xy\,dy + y^2\,dx - 3dx + 2dy &= 0 \\ (2xy + 2)dy &= (3 - y^2 - 2x)dx \\ dy &= \frac{(3 - y^2 - 2x)dx}{2(xy + 1)}. \end{aligned}$$

4. If $dy = x\,dx$,

$$\begin{aligned} y &= \int x\,dx \\ &= \tfrac{1}{2}\int 2x\,dx \\ &= \frac{x^2}{2} + C. \end{aligned}$$

5. If $dy = x\sqrt{1 - x^2}\,dx$,

$$\begin{aligned} y &= \int x(1 - x^2)^{\frac{1}{2}}\,dx \\ &= -\tfrac{1}{2}\,\tfrac{2}{3}\int \tfrac{3}{2}(1 - x^2)^{\frac{1}{2}}(-2x\,dx) \\ &= -\frac{(1 - x^2)^{\frac{3}{2}}}{3} + C. \end{aligned}$$

6. If $\dfrac{dy}{y} = \dfrac{dx}{x - 1}$,

$$y = C(x - 1).$$

by formula 10.

7. If $\dfrac{dy}{y} = \dfrac{x\,dx}{x^2 - 1}$,

$$\frac{dy}{y} = \frac{1}{2}\frac{2x\,dx}{x^2 - 1}$$
$$y = C\sqrt{x^2 - 1}.$$

Exercises

Find dy in each of the following sixteen exercises:

1. $y = x^2 - 3x - 2.$
2. $y = x^3 - x^2 - x + 1.$
3. $y = (x^2 + 2x - 2)^3.$
4. $y = (x^2 + x + 1)^4.$
5. $y = (x - 2)^{\frac{1}{2}}.$
6. $y = (x^2 - 2)^{\frac{1}{3}}.$
7. $y = \dfrac{x}{x - 1}.$
8. $y = \dfrac{x^{\frac{1}{2}}}{1 - x^2}.$
9. $y = (x - 1)^{-\frac{1}{2}}.$
10. $y = (x^2 - 1)^{-\frac{1}{2}}.$
11. $xy - x + y = 2.$
12. $y^2 - xy + x^2 = 3.$
13. $x^2 - y^2 = a^2.$
14. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$
15. $y = \dfrac{\sqrt{x}}{(x - 1)^2}.$
16. $y = \dfrac{x}{\sqrt{1 - x^2}}.$

Integrate the following:

17. $\displaystyle\int x(x^2 - 1)dx.$
18. $\displaystyle\int x(x^2 - 1)^2dx.$
19. $\displaystyle\int (x^3 - 3x)(x^2 - 1)dx.$
20. $\displaystyle\int (x^2 - 2x + 5)^2(x - 1)dx.$
21. $\displaystyle\int \frac{dx}{x^2}.$
22. $\displaystyle\int \frac{x\,dx}{(x^2 - 1)^2}.$
23. $\displaystyle\int \sqrt{x}\,dx.$
24. $\displaystyle\int (x^3 - 3x^2 + 3x)^3(x^2 - 2x + 1)dx.$
25. $\displaystyle\int \frac{(x - 1)dx}{(x^2 - 2x + 3)^2}.$
26. $\displaystyle\int \frac{(x + 1)dx}{\sqrt{x^2 + 2x - 5}}.$
27. $\displaystyle\int (x - 1)(x^2 - 2x + 7)^{-\frac{1}{3}}dx.$
28. $\displaystyle\int (x - 2)(x^2 - 4x - 7)^{-\frac{3}{2}}dx.$

62. Approximations. In §60 it was shown that the difference $dy - \Delta y$ is an infinitesimal of higher order than Δy. Hence in many cases it is possible to use dy, which is readily calculated, as a sufficiently close approximation to Δy.

Recalling that

$$\Delta y = dy + \epsilon\,\Delta x,$$

or

$$\frac{\Delta y - dy}{\Delta y} = \epsilon\,\frac{\Delta x}{\Delta y}.$$

where ϵ approaches zero with Δx, we see that the relative error with respect to Δy, made by using dy instead of Δy, approaches zero as Δx approaches zero.

Illustration 1. The edge of a cube is measured and found to be 10.2 inches long. We must, however, allow for a possible error, say 0.1 inch, in the measurement of the true length of the edge. Our measurement may be too large or too small. But with the measuring instruments used, we have reason to believe that the magnitude of the error does not exceed 0.1 inch. It is desired to determine approximately the maximum error possible in the volume calculated by using the measured length of the edge.

The volume of the cube as calculated from our measurement is $(10.2)^3$, or 1061.208 cubic inches. But since the measured length may be in error, this calculated volume is very probably not the true volume.

Let y denote the volume and x the edge of the cube. Then

$$y = x^3$$

and

$$dy = 3x^2\,dx,$$

where $dx(=\Delta x)$ is ± 0.1, the assumed maximum error in the measured length and $x = 10.2$. We find $dy = \pm 31.2$ which is a close approximation to the value of Δy corresponding to $\Delta x = \pm 0.1$. The error in the volume is not greater than 31.2 cubic inches approximately. That is, the volume lies between the approximate limits $1061.2 - 31.2 = 1030.0$ cubic inches and $1061.2 + 31.2 = 1092.4$ cubic inches. Exact calculations show

that the volume lies between 1030.3 and 1092.7 cubic inches. There is thus a difference of about 3 parts in 10,000 in the limits of the volume as given by the approximate and exact methods.

The approximate percentage error in the volume can be calculated in terms of the percentage error of the edge by formula (10), §61. Thus

$$\frac{dy}{y} = 3\frac{dx}{x}.$$

This formula shows that the approximate percentage error of the volume is equal to three times the percentage error of the edge. The reader should verify this conclusion for the example under consideration above.

Illustration 2. The diameter of a circle is approximately 25 inches. How accurately must the diameter of the circle be measured in order that the area of the circle can be calculated with an error not greater than 2 per cent?

Let y be the area and x the diameter of the circle. Then

$$y = \frac{\pi}{4}x^2$$

and

$$\frac{dy}{y} = 2\frac{dx}{x},$$

or

$$\frac{dx}{x} = \tfrac{1}{2}\left(\frac{dy}{y}\right) = \tfrac{1}{2}(0.02) = 0.01.$$

Thus the diameter must be measured with an error not greater than 1 per cent, or not greater than 0.25 inch.

Exercises

1. The radius of a circle is measured and found to be 8.2 inches with a possible error of 0.05 inch. How accurately can the area of the circle be found from this measurement?

2. The side of a square field is measured and found to be 159.3 rods with a possible error of 0.02 rod. How accurately can the area of the field be found from this measurement?

3. The area of a circle is to be found with an error not greater than 3 per cent. How accurately must the diameter be measured?

4. The volume of a sphere is to be found with an error not greater than 2 per cent. How accurately must the diameter be measured?

5. The surface of a sphere is to be found with an error not greater than 3 per cent. How accurately must the diameter be measured?

6. The side of an equilateral triangle is measured and found to be 11.2 cm. with a possible error not greater than 0.5 per cent. Find the possible percentage error in the calculated area of the triangle.

7. The base and altitude of a triangle are measured and found to be, respectively, 12.4 centimeters and 15.6 centimeters. Assuming a possible maximum error of 0.05 centimeter and 0.01 centimeter in the measurements of the base and altitude, respectively, find the possible error in the area as calculated from these measurements.

8. Same as Exercise 7, but let the polygon be a parallelogram. Compare the results of the two exercises if each is expressed in terms of percentage error.

9. A clock gains 2 minutes a day. The period of a half swing of the pendulum is intended to be 1 second. If the period of the half swing of the pendulum is given by

$$T = \pi\sqrt{\frac{l}{g}},$$

where l is the length of the pendulum in centimeters and $g = 980$, find the error in the length of the pendulum.

10. The formula for the flow of water over a certain type of weir is $Q = 3.37LH^{\frac{3}{2}}$, where L is the length of the crest measured in feet, H the height of the water above the crest measured in feet, and Q the quantity of water flowing over the weir measured in cubic feet per second. Find the possible error in the calculated value of Q if $L = 1.5 \pm 0.001$ and $H = 0.932 \pm 0.003$. The numbers following the plus and minus signs in the values of L and H represent the possible errors in the respective measurements.

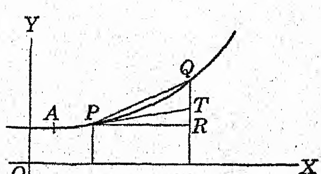

FIG. 38.

63. The Limit of the Ratio of the Arc to the Chord. Let P, Fig. 38, be any point on the curve $y = f(x)$. Let $PR = \Delta x$, $RQ = \Delta y$, the chord $PQ = c$, and the arc

$PQ = \Delta s$. (s represents the length of arc measured from some point A.) PT is the tangent at P.

When Δx is taken so small that the curve has no point of inflection between P and Q, the chord $PQ <$ arc $PQ < PT + TQ$, or $c < \Delta s < PT + TQ$. Whence,

$$1 < \frac{\Delta s}{c} < \frac{PT}{c} + \frac{TQ}{c}. \tag{1}$$

$$\left(\frac{PT}{c}\right)^2 = \frac{(dx)^2 + (dy)^2}{(\Delta x)^2 + (\Delta y)^2} = \frac{1 + \left(\frac{dy}{dx}\right)^2}{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

Therefore

$$\lim_{\Delta x \to 0} \left(\frac{PT}{c}\right)^2 = 1.$$

$$\lim_{\Delta x \to 0} \frac{TQ}{c} = \lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta y} \frac{\Delta y}{c}$$

$$= \left[\lim_{\Delta x \to 0}\left(1 - \frac{dy}{\Delta y}\right)\right]\left[\lim_{\Delta x \to 0} \frac{\Delta y}{c}\right] = 0,$$

since

$$\lim_{\Delta x \to 0} \frac{dy}{\Delta y} = 1.$$

Then from (1),

$$\lim_{\Delta x \to 0} \frac{\Delta s}{c} = 1.$$

64. Differential of Length of Arc, Rectangular Coordinates. From Fig 38 we have

$$(c)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\lim_{\Delta x \to 0} \left(\frac{c}{\Delta x}\right)^2 = 1 + \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

Since

$$\lim_{\Delta x \to 0} \frac{c}{\Delta s} = 1,$$

c can be replaced by Δs, §59.

$$\lim_{\Delta x \to 0} \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2,$$

or

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

$$(\mathbf{ds})^2 = (\mathbf{dx})^2 + (\mathbf{dy})^2, \tag{1}$$

$$\mathbf{ds} = \sqrt{1 + \left(\frac{\mathbf{dy}}{\mathbf{dx}}\right)^2}\,\mathbf{dx}, \tag{2}$$

$$\mathbf{ds} = \sqrt{1 + \left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^2}\,\mathbf{dy}. \tag{3}$$

Equation (1) shows that the line PT, Fig. 38, represents ds. The right triangle PRT with hypotenuse ds and legs dx and dy should be used to recall equation (1) from which equations (2) and (3) can be readily obtained. It is to be noted that

$$\lim_{\Delta x \to 0} \frac{ds}{\Delta s} = \lim_{\Delta x \to 0} \frac{PT}{\text{arc } PQ} = 1.$$

Also that

$$\lim_{\Delta x \to 0} \frac{ds}{c} = \lim_{\Delta x \to 0} \frac{PT}{\text{chord } PQ} = 1.$$

In other words, these three infinitesimals, Δs, ds, and c, differ from one another by infinitesimals of higher order. If τ denotes the angle made by the line PT with the positive X-axis,

$$\mathbf{dx} = \cos \tau \; \mathbf{ds}$$
$$\mathbf{dy} = \sin \tau \; \mathbf{ds}.$$

Let a particle move along the curve of Fig. 38. When it is at the point P, its velocity is represented by a vector of magnitude $\frac{ds}{dt}$ directed along the tangent PT. From equation (1)

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \tag{4}$$

Thus the velocity at any instant is the resultant of the two component velocities, $v_x = \frac{dx}{dt}$ and $v_y = \frac{dy}{dt}$, parallel to the axes of x and y, respectively.

Illustration 1. Find the length of the portion of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ which lies above the X-axis between the lines $x = 3$ and $x = 8$.

$$\frac{dy}{dx} = x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = x.$$

Substituting in formula (2),

$$ds = \sqrt{1 + x}\, dx.$$

Integrating,

$$s = \tfrac{2}{3}(1 + x)^{\frac{3}{2}} + C.$$

When $x = 3$, $s = 0$. Hence $C = -\frac{16}{3}$, and

$$s = \tfrac{2}{3}(1 + x)^{\frac{3}{2}} - \tfrac{16}{3}.$$

This formula gives the length of the curve measured from the point whose abscissa is 3 to the point whose abscissa is x. On placing $x = 8$ we obtain $s = \frac{38}{3}$, the length of the curve from the point corresponding to $x = 3$ to the point corresponding to $x = 8$.

Illustration 2. A particle moves along the parabola $y^2 = x$, for which the unit of length is 1 foot. The component of its velocity parallel to the axis of x is constant and equal to 2 feet per second. Find the magnitude and direction of the velocity of the particle at any instant, and in particular at the instant when $x = 3$. Using implicit differentiation,

$$2y\frac{dy}{dt} = \frac{dx}{dt}.$$

$$\frac{dy}{dt} = \frac{1}{2y}\frac{dx}{dt} = \pm\frac{1}{2\sqrt{x}}\frac{dx}{dt}.$$

For positions above the X-axis the positive sign of the radical is to be used.

Since

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = \frac{1}{\sqrt{x}}.$$

Hence by (4),

$$\frac{ds}{dt} = \sqrt{4 + \frac{1}{x}} = \sqrt{\frac{4x + 1}{x}}.$$

When

$$x = 3, \frac{ds}{dt} = \sqrt{\frac{13}{3}} = 2.08.$$

Thus the magnitude of the velocity is 2.08 feet per second correct to three significant figures. The tangent of the angle τ between the direction of the velocity and the positive direction of the X-axis is given by

$$\tan \tau = \frac{v_y}{v_x} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{2\sqrt{x}}.$$

When

$$x = 3, \tan \tau = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}.$$

Illustration 3. The coordinates of a moving particle are given in terms of the time by the equations $x = 2t$, $y = t^2$. Find the velocity at any instant.

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 2t.$$

By (4),

$$\frac{ds}{dt} = \sqrt{4 + 4t^2} = 2\sqrt{1 + t^2}.$$

If τ is the angle which the vector representing the velocity makes with the X-axis,

$$\tan \tau = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t.$$

Exercises

1. Find the length of the portion of the curve $y = x^{\frac{3}{2}}$ which lies above the X-axis between the points whose abscissas are 0 and 4.

Find the differential of the length of arc of the three following curves:

2. $y^2 = x$. **3.** $xy = 1$. **4.** $y = x^3$.

5. A particle moves on the circle $x^2 + y^2 = 9$. At the point $(2\sqrt{2}, 1)$ the horizontal component of its velocity is 1 foot per second. Find its velocity in magnitude and direction at this point.

6. A particle moves on the parabola $y^2 = x$ at a uniform speed of 3 feet per second. Find the horizontal and vertical components of its velocity when $x = 6$.

7. A particle moves on the circle $x^2 + y^2 = 100$ at the uniform speed of 5 feet per second. At what rate is its projection on the X-axis moving when $x = 6$?

8. The path of a moving particle is given by the equations $x = t^2$, $y = t^3$, t being the time. Find the velocity of the particle at any instant.

9. Find the length of the curve of Exercise 8 between the points corresponding to $t = 1$ and $t = 5$. Hint. Use $ds = \sqrt{(dx^2) + (dy)^2}$.

10. A toy engine is running on an elliptical track whose axes are 10 and 6 feet, respectively, at a uniform speed of 2 feet per second. The longer axis of the ellipse is parallel to a wall of the room. There is a distant light whose rays may be considered perpendicular to the wall. Find the rate at which the shadow of the engine is moving along the wall when the engine is 3 feet from the minor axis of the ellipse.

65. The Limit of $\Sigma f(x)\Delta x$. Let $f(x)$ be a function which is continuous in the interval $a \leqq x \leqq b$. In §51 it was shown that the area bounded by the curve, the X-axis, and the ordinates $x = a$

and $x = b$ is given by the formula

$$A = F(b) - F(a), \tag{1}$$

where $F(x) = \int f(x)dx$. A second expression will now be found for the area. Divide the interval $b - a$ on the X-axis, Fig. 39, into n equal parts and at each point of division erect an ordinate. Complete the rectangles as indicated in the figure. It is assumed at first that $f(x)$ is an increasing function throughout the interval

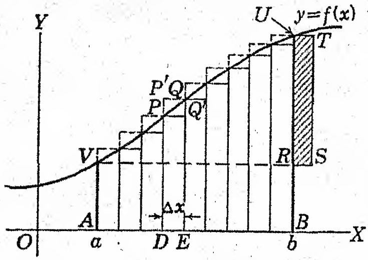

FIG. 39.

from $x = a$ to $x = b$. If it is an increasing function in some portions of the interval and a decreasing function in others, the proof that follows is to be applied separately to each of the portions indicated.

The sum of the rectangles, of which $DEQ'P$ is a type, is approximately equal to the area $ABUV$. The greater n, the number of rectangles, *i.e.*, the smaller Δx, the closer will the sum of the rectangles approximate the area $ABUV$. We say then that

$$A = \lim_{n \to \infty} \Sigma\, DEQ'P,$$

or

$$A = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x)\Delta x. \tag{2}$$

The expression (2) represents the *actual* area and *not* an approximation to it, as can be shown by finding the greatest possible error corresponding to a given number of rectangles and then proving that this error approaches zero as the number of rectangles

becomes infinite. Thus it is easily seen that the difference between the area A and the sum of the rectangles $DEQ'P$ is less than the area of the rectangle $RSTU$. The altitude, $f(b) - f(a)$, of this rectangle is constant while the length of the base, Δx, approaches zero. Hence the area of $RSTU$ approaches zero as n increases without limit, and consequently the difference between the area A and the sum of the rectangles $DEQ'P$ can be made as small as we please in numerical value by taking n sufficiently large. Therefore the limit of the sum of the rectangles is the area sought.

On equating the two expressions for A, given by (1) and (2), we have

$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x)\Delta x = F(b) - F(a), \tag{3}$$

where

$$F(x) = \int f(x)dx.$$

This equation is *the important result* of this section. It gives a means of calculating

$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x)\Delta x,$$

by finding the integral of $f(x)dx$ and taking the difference between the values of this integral at $x = a$ and $x = b$. The result of this section will be restated and emphasized in the next section.

66. Definite Integral. The expression

$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x)\Delta x$$

which was introduced in the preceding section is of such great importance that it is given a name, "*the definite integral of $f(x)$ between the limits a and b,*" *and is denoted by the symbol*

$$\int_a^b f(x)dx.$$

Equation (3), §65, gives a means of calculating the value of the definite integral.

The function $F(x)$, the integral of $f(x)dx$, is called the *indefinite integral* of $f(x)dx$ in order to distinguish it from the definite integral which is defined independently of it, *viz.*, as the limit of a certain sum.

We have then the following definition and theorem:

Definition. *Let $f(x)$ be a continuous function in the interval from* $x = a$ to $x = b$, *and let this interval be divided into n equal parts of length Δx by points $x_1, x_2, x_3, \ldots, x_{n-1}$. The "definite integral of $f(x)$ between the limits a and b" is the limit of the sum of the products $f(x_i)\,\Delta x$ formed for all of the points $x_0 = a, x_1, x_2, \ldots, x_{n-1}$, as the number of divisions becomes infinite.*

Theorem.—*The definite integral of $f(x)$ between the limits a and b is calculated by finding the indefinite integral, $F(x)$ of $f(x)dx$ and forming the difference $F(b) - F(a)$.*

The symbol for the definite integral,

$$\int_a^b f(x)dx,$$

is read "the integral from a to b of $f(x)dx$." As noted above, this symbol represents

$$\lim_{\Delta x \to 0} \sum_a^b f(x)\Delta x.$$

Many problems, such as finding the work done by a variable force, the volume of a solid, the coordinates of the center of gravity, lead to definite integrals. But no matter how a definite integral may have been obtained and no matter what other meaning it may have, it may always be regarded as representing the area included by the curve $y = f(x)$, the X-axis, and the ordinates $x = a$ and $x = b$, provided that $f(x)$ is a function which can be represented by a continuous curve. The fact, that

$$\int_a^b f(x)dx$$

becomes infinite. Thus it is easily seen that the difference between the area A and the sum of the rectangles $DEQ'P$ is less than the area of the rectangle $RSTU$. The altitude, $f(b) - f(a)$, of this rectangle is constant while the length of the base, Δx, approaches zero. Hence the area of $RSTU$ approaches zero as n increases without limit, and consequently the difference between the area A and the sum of the rectangles $DEQ'P$ can be made as small as we please in numerical value by taking n sufficiently large. Therefore the limit of the sum of the rectangles is the area sought.

On equating the two expressions for A, given by (1) and (2), we have

$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x)\Delta x = F(b) - F(a), \tag{3}$$

where

$$F(x) = \int f(x)dx.$$

This equation is *the important result* of this section. It gives a means of calculating

$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x)\Delta x,$$

by finding the integral of $f(x)dx$ and taking the difference between the values of this integral at $x = a$ and $x = b$. The result of this section will be restated and emphasized in the next section.

66. Definite Integral. The expression

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Equation (3), §**65**, gives a means of calculating the value of the definite integral.

The function $F(x)$, the integral of $f(x)dx$, is called the *indefinite integral* of $f(x)dx$ in order to distinguish it from the definite integral which is defined independently of it, *viz.*, as the limit of a certain sum.

We have then the following definition and theorem:

Definition. *Let $f(x)$ be a continuous function in the interval from $x = a$ to $x = b$, and let this interval be divided into n equal parts of length Δx by points $x_1, x_2, x_3, \ldots, x_{n-1}$. The "definite integral of $f(x)$ between the limits a and b" is the limit of the sum of the products $f(x_i)\,\Delta x$ formed for all of the points $x_0 = a, x_1, x_2, \ldots, x_{n-1}$, as the number of divisions becomes infinite.*

Theorem.—*The definite integral of $f(x)$ between the limits a and b is calculated by finding the indefinite integral, $F(x)$ of $f(x)dx$ and forming the difference $F(b) - F(a)$.*

The symbol for the definite integral,

$$\int_a^b f(x)dx,$$

is read "the integral from a to b of $f(x)dx$." As noted above, this symbol represents

$$\lim_{\Delta x \to 0} \sum_a^b f(x)\Delta x.$$

Many problems, such as finding the work done by a variable force, the volume of a solid, the coordinates of the center of gravity, lead to definite integrals. But no matter how a definite integral may have been obtained and no matter what other meaning it may have, it may always be regarded as representing the area included by the curve $y = f(x)$, the X-axis, and the ordinates $x = a$ and $x = b$, provided that $f(x)$ is a function which can be represented by a continuous curve. The fact, that

$$\int_a^b f(x)dx$$

can be regarded as representing an area, enables us to calculate its value. For the area in question is equal to $F(b) - F(a)$, where $F(x)$ is the indefinite integral of $f(x)dx$. Consequently, we have, in all cases,

$$\int_a^b \mathbf{f(x)dx = F(b) - F(a).} \tag{1}$$

This is often written

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a),$$

to show how the result is to be calculated. Thus

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}.$$

It is to be noted that

$$\int_b^a f(x)dx = F(a) - F(b) = -\int_a^b f(x)dx. \tag{2}$$

Exercises

Evaluate each of the following ten definite integrals:

1. $\displaystyle\int_2^4 (2x + 3)dx.$
2. $\displaystyle\int_2^4 \frac{dx}{x^2}.$
3. $\displaystyle\int_0^a x\sqrt{a^2 - x^2}\,dx.$
4. $\displaystyle\int_{-1}^1 (1 - x^2)dx.$
5. $\displaystyle\int_1^2 \frac{x\,dx}{\sqrt{9 - x^2}}.$
6. $\displaystyle\int_0^3 \frac{x\,dx}{(1 + x^2)^2}.$
7. $\displaystyle\int_0^2 x\sqrt{4 - x^2}\,dx.$
8. $\displaystyle\int_2^3 (1 + x^2)^2dx.$
9. $\displaystyle\int_1^2 \frac{x - 1}{x^3}\,dx.$
10. $\displaystyle\int_{-1}^0 \sqrt{1 - x}\,dx.$

In each of Exercises 11 to 16 set up and evaluate the definite integral representing the area bounded by the curve and the X-axis.

11. $y = 16 - x^2$.

12. $y = 2x - x^2$.

13. $(1 - y)^3 = x^2$.

14. $(4 - y)^3 = 4(x - 1)^2$.

15. $y + 2x^2 + 12x + 16 = 0$.

16. $y = 1 - x^4$.

17. Find, by means of definite integrals, the area completely bounded by the curves:

$$2(y - 3) = -(x - 1)^2 \quad \text{and} \quad y + x = 4.$$

67. Duhamel's Theorem. *If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are n infinitesimals of like sign, the limit of whose sum is finite as n becomes infinite, and if $\beta_1, \beta_2, \beta_3, \ldots, \beta_n$ are a second set of infinitesimals such that*

$$\lim_{n \to \infty} \frac{\beta_i}{\alpha_i} = 1,$$

where $i = 1, 2, 3, \cdots, n$, then

$$\lim_{n \to \infty} \sum_1^n \alpha_i = \lim_{n \to \infty} \sum_1^n \beta_i.$$

Proof. Let

$$\frac{\beta_i}{\alpha_i} = 1 + \epsilon_i.$$

Since

$$\lim_{n \to \infty} \frac{\beta_i}{\alpha_i} = 1,$$
$$\lim_{n \to \infty} \epsilon_i = 0.$$

At first let it be assumed that the α's are positive. Let E be the numerical value of the largest ϵ, *i.e.*,

$$E \geqq |\epsilon_i|, \qquad i = 1, 2, 3, \cdots, n.$$

Then, since $\beta_i = \alpha_i + \epsilon_i\alpha_i, \quad i = 1, 2, 3, \cdots, n$,

$$\alpha_1 - E\alpha_1 \leqq \beta_1 \leqq \alpha_1 + E\alpha_1$$
$$\alpha_2 - E\alpha_2 \leqq \beta_2 \leqq \alpha_2 + E\alpha_2$$
$$\cdots\cdots\cdots$$
$$\alpha_n - E\alpha_n \leqq \beta_n \leqq \alpha_n + E\alpha_n.$$

Adding, we get

$$(1 - E)\sum_{i=1}^{i=n}\alpha_i \leqq \sum_{i=1}^{i=n}\beta_i \leqq (1 + E)\sum_{i=1}^{i=n}\alpha_i.$$

Since

$$\lim_{n \to \infty} E = 0,$$

$$\lim \sum_{i=1}^{i=n}\alpha_i = \lim \sum_{i=1}^{i=n}\beta_i$$

and the theorem is proved.

If the α's are negative, it will be necessary to change the proof just given, only by reversing the signs of inequality.

Section **65** furnishes an illustration of this theorem. In this example the limit of the sum of the infinitesimal trapezoidal areas $DEQP$ is finite as n becomes infinite, since it is the area sought.

$$DEQ'P < DEQP < DEQP',$$

(see Fig. 39), or

$$y\Delta x < DEQP < (y + \Delta y)\Delta x,$$

or

$$1 < \frac{DEQP}{DEQ'P} < \frac{y + \Delta y}{y}.$$

This shows that the limit of the ratio of the trapezoidal area to the area of the corresponding rectangle is 1 as n becomes infinite. Then, by Duhamel's theorem,

$$\lim_{n \to \infty} \Sigma\, DEQ'P = \lim_{n \to \infty} \Sigma\, DEQP = A.$$

Since we are able to replace the infinitesimals $DEQP$ by the infinitesimals $DEQ'P$, we may express the area, which is the limit of the sum of the infinitesimals $DEQP$, by the limit of the sum of the

infinitesimals $DEQ'P$, *i.e.*, by the definite integral $\lim_{\Delta x \to 0} \sum_a^b f(x)\Delta x$.

This is a characteristic process in the use of the definite integral. The quantity sought is subdivided into n portions which are infinitesimals as n becomes infinite. These are replaced by n other infinitesimals of the form $f(x_i)\Delta x$. The limit of the sum of the latter infinitesimals is a definite integral.

Illustrations of the applications of Duhamel's theorem to obtain definite integrals representing work, force, volume, etc. follow.

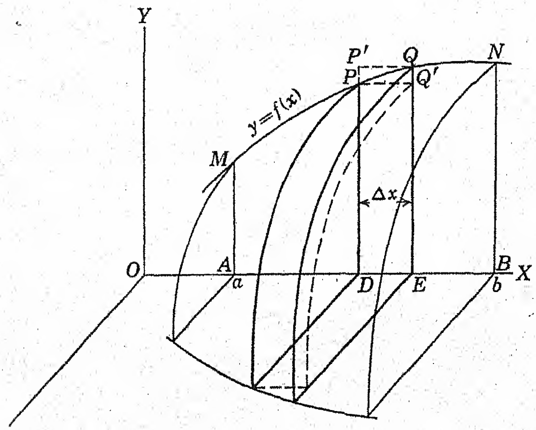

Fig. 40.

68. Volume of a Solid of Revolution. The area bounded by the curve $y = f(x)$, Fig. 40, the ordinates $x = a$ and $x = b$, and the X-axis is revolved about the X-axis. Find the volume of the solid generated. Only one-fourth of the volume generated is shown in Fig. 40.

Divide the interval $AB = b - a$ on the X-axis into n equal parts of length Δx and pass planes through the points of division perpendicular to the X-axis. These planes divide the volume into n portions, ΔV. A typical portion can be regarded as generated by revolving $DEQP$, Fig. 40, about the base DE in the X-axis.

If V denotes the volume sought, then

$$V = \sum_{x=a}^{x=b} \Delta V.$$

This relation holds no matter how large n is taken. Hence

$$V = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \Delta V. \tag{1}$$

Now ΔV, the volume generated by revolving $DEQP$ in Fig. 40 about the X-axis, is greater than that of the cylinder generated by the revolution of the rectangle $DEQ'P$ and less than that of the cylinder generated by the revolution of the rectangle $DEQP'$. That is, denoting DP by y and EQ by $y + \Delta y$,

$$\pi y^2 \Delta x < \Delta V < \pi(y + \Delta y)^2 \Delta x.$$

Then

$$1 < \frac{\Delta V}{\pi y^2 \Delta x} < \frac{(y + \Delta y)^2}{y^2}.$$

Since Δy approaches zero as Δx approaches zero, it follows that

$$\lim_{\Delta x \to 0} \frac{\Delta V}{\pi y^2 \Delta x} = 1. \tag{2}$$

This relation holds for each of the infinitesimals in equation (1). Consequently, in accordance with Duhamel's theorem,

$$V = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \Delta V = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \Delta x = \int_a^b \pi y^2 \, dx. \tag{3}$$

We have thus replaced the expression (1), which cannot be calculated directly, by an equivalent expression, the definite integral in (3), whose value can usually be readily calculated. With the aid of Duhamel's theorem we have replaced a set of infinitesimals the limit of whose sum is difficult to calculate by another set of

infinitesimals, the limit of whose sum is expressed by a simple definite integral.

Illustration. Find the volume between the planes $x = 1$ and $x = 3$ of the solid generated by revolving the curve $y = x^2 + x$ about the X-axis.

$$V = \lim_{\Delta x \to 0} \sum_a^b \pi y^2 \Delta x = \pi \int_1^3 y^2 dx = \pi \int_1^3 (x^4 + 2x^3 + x^2) dx$$

$$= \pi[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3]_1^3 = \pi x^3[\tfrac{1}{5}x^2 + \tfrac{1}{2}x + \tfrac{1}{3}]_1^3$$

$$= 27\pi(\tfrac{9}{5} + \tfrac{3}{2} + \tfrac{1}{3}) - \pi(\tfrac{1}{5} + \tfrac{1}{2} + \tfrac{1}{3}) = \tfrac{1456}{15}\pi.$$

Exercises

1. Find the volume of the solid generated by revolving about the X-axis the portion of the curve $y^2 = x$ which lies between $x = 1$ and $x = 4$.

2. Find the volume of the frustum of a right circular cone generated by revolving about the X-axis the portion of $2y = x + 6$ which lies between $x = 0$ and $x = 4$.

3. Find the volume of the solid generated by revolving about the X-axis the portion of the line through $(0, 0)$ and (h, r) which lies between $x = 0$ and $x = h$.

4. Find the volume of the solid generated by revolving about the X-axis the portion of $y = 4 - x^2$ which lies above the X-axis.

5. Find by the method of this section the volume of a sphere of radius a.

6. Find the volume of the solid generated by revolving the hypocycloid, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, about the X-axis.

7. Find the volume of the ellipsoid generated by revolving

$$16x^2 + 25y^2 = 400$$

about the X-axis. About the Y-axis.

8. Same as Exercise 7 but use the equation $b^2x^2 + a^2y^2 = a^2b^2$.

9. Find the volume of the solid generated by revolving about the X-axis the portion of the curve $y = 9 - x^2$ which lies above the X-axis. Find the volume generated by revolving the same portion of this curve about the Y-axis.

10. Find the volume of the solid generated by revolving about the X-axis the portion of the curve $y = x^2 + 2$ which lies between the lines $x = -2$ and $x = 2$.

11. Find the volume of the solid generated by revolving about the X-axis the portion of the curve $y = x^{\frac{3}{2}}$ which lies between the lines $x = 0$ and $x = 4$.

12. The area between the parabola $y = \dfrac{x^2}{25} + 2$, the X-axis, and the lines $x = -5$ and $x = 5$ is revolved about the Y-axis. Find the volume of the solid generated.

13. The area between the hyperbola $x^2 - y^2 = 9$ and the lines $y = -4$ and $y = 4$ is revolved about the X-axis. Find the volume of the solid generated.

14. Find the volume of a frustum of a cone, denoting the altitude by h and the radii of the bases by R and r.

15. Find the volume generated by revolving the loop of $y^2 = x(x-1)(x-2)$ about the X-axis.

69. Work Done by a Variable Force. In §52 there was found the work done by a variable force $f(s)$, in producing a displacement from $s = a$ to $s = b$. We shall now obtain the same result by building up the definite integral which represents the work.

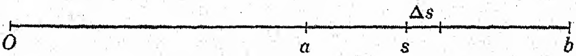

FIG. 41.

Divide the total displacement $b - a$, Fig. 41, into n equal parts of length Δs. The force acting at the left end of one of these parts is $f(s)$, while that acting at the right end is $f(s + \Delta s)$. The total work done in producing the displacement, $b - a$, is approximately

$$\sum_{s=a}^{s=b} f(s)\Delta s.$$

The actual work is the limit of this sum as Δs approaches zero, *viz.*,[1]

[1] This step can be justified by using Duhamel's theorem. Let Δw represent the work done in producing the displacement Δs. Then

$$w = \lim_{\Delta s \to 0} \sum_{a}^{b} f(s)\Delta s = \int_a^b f(s)ds.$$

Illustration 1. The solution of the problem of *Illustration* 1, **§52,** is expressed by

$$w = \lim_{\Delta s \to 0} \sum_{s=1}^{s=4} 50s\Delta s = \int_1^4 50sds = 25s^2 \Big|_1^4 = 375.$$

Illustration 2. The solution of the problem of *Illustration* 3, **§52,** is expressed by

$$w = \lim_{\Delta s \to 0} \sum_{s=a}^{s=b} \frac{kmM}{s^2}\Delta s = kmM \int_a^b \frac{ds}{s^2} = -kmM\frac{1}{s}\Big|_a^b$$

$$= -kmM\left[\frac{1}{b} - \frac{1}{a}\right] = kmM\left[\frac{1}{a} - \frac{1}{b}\right].$$

Illustration 3. In solving the problem of *Illustration* 2, **§52,** we can write, if s_1 and s_2 are the values of s corresponding to $v = 3$ and $v = 4$,

$$w = \lim_{\Delta s \to 0} \sum_{s=s_1}^{s=s_2} (pA)\Delta s = \lim_{\Delta s \to 0} \sum_{s=s_1}^{s=s_2} pA\Delta s$$

$$= \lim_{\Delta v \to 0} \sum_{v=3}^{v=4} p\Delta v = \int_3^4 pdv.$$

$$W = \lim_{n \to \infty} \Sigma \Delta w.$$

But

$$(s)\Delta s < \Delta w < f(s + \Delta s)\Delta s,$$

or

$$1 < \frac{\Delta w}{f(s)\Delta s} < \frac{f(s + \Delta s)}{f(s)}.$$

Then

$$\lim_{\Delta s \to 0} \frac{\Delta w}{f(s)\Delta s} = 1.$$

Hence, by Duhamel's theorem,

$$\lim_{\Delta s \to 0} \sum_{s=a}^{s=b} \Delta w = \lim_{\Delta s \to 0} \sum_{a}^{b} f(s)\Delta s = \int_a^b f(s)ds.$$

Since $pv^{1.4} = C$, $p = \frac{C}{v^{1.4}}$, and

$$w = C\int_3^4 \frac{dv}{v^{1.4}} = -\frac{C}{0.4}\frac{1}{v^{0.4}}\Big|_3^4 = -\frac{C}{0.4}\left[\frac{1}{4^{0.4}} - \frac{1}{3^{0.4}}\right].$$

The numerical calculation was performed in §52.

Illustration 4. Water is pumped from a round cistern whose median section is a semi-ellipse, whose minor axis, 8 feet long, is a

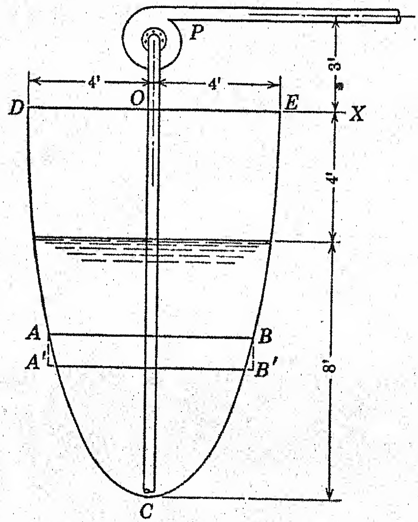

FIG. 42.

diameter of the top of the cistern. The cistern is 12 feet deep and the water is 8 feet deep. Find the work done in pumping the water from the cistern if the discharge of the pump is 3 feet above the top of the cistern and if the friction in the pump and the friction of the water in the pipes are neglected.

In Fig. 42, let $DACBE$ represent a median section of the cistern, and let P represent the position of the pump. The equation of the ellipse is $9x^2 + y^2 = 144$. For convenience y is taken to be positive when measured downward.

Through the volume of water in the cistern pass horizontal planes Δy units apart. Let AB represent a typical plane passing through the point (x, y) of the median section and $A'B'$ a plane passing through the point $(x + \Delta x,\ y + \Delta y)$. The weight of water lying between these two planes is approximately $k\pi x^2\Delta y$ pounds, where k is the weight of a cubic foot of water. (Use $k = 62.5$ pounds.) The distance through which this weight of water is lifted is approximately $(3 + y)$ feet. Hence the work done in lifting it is approximately $k\pi x^2\Delta y(3 + y)$ foot-pounds. The total work W done in emptying the cistern is then

$$W = \lim_{\Delta y \to 0} \sum_{y=4}^{y=12} k\pi x^2(3 + y)\Delta y$$

$$= k\pi \int_4^{12} x^2(3 + y)dy = \frac{k\pi}{9} \int_4^{12} (144 - y^2)(3 + y)dy. \quad (1)$$

To carry out the integration, form the product of the factors in the integrand and integrate each term. It will be found that $W =$ 106,680 foot-pounds.

The steps taken in setting up the integral can be justified by the use of Duhamel's theorem as follows: Let Δw denote the work done in lifting the water between two typical planes AB and $A'B'$. Then

$$W = \sum_{y=4}^{y=12} \Delta W = \lim_{\Delta y \to 0} \sum_{y=4}^{y=12} \Delta W \quad (2)$$

since this relation holds no matter how small Δy is taken. But

$$k\pi(x + \Delta x)^2\Delta y(3 + y) < \Delta W < k\pi x^2\Delta y(3 + y + \Delta y). \quad (3)$$

After dividing each member of (3) by $k\pi x^2\Delta y(3 + y)$, it follows readily that

$$\lim_{\Delta y \to 0} \frac{\Delta W}{k\pi x^2\Delta y(3 + y)} = 1.$$

Hence, in accordance with Duhamel's theorem,

$$W = \lim_{\Delta y \to 0} \sum_{y=4}^{y=12} \Delta W = \lim_{\Delta y \to 0} \sum_{y=4}^{y=12} k\pi x^2 \Delta y(3 + y)$$

$$= k\pi \int_4^{12} x^2(3 + y)dy.$$

Exercises

1. Set up and evaluate the definite integrals representing the work sought in Exercises 1, 2, and 6, §**52**.

2. Water is pumped from a round cistern as in *Illustration* 4. The median section is a parabola. The diameter of the top of the cistern is 6 feet. The depth of the cistern is 14 feet. Find the work done in pumping the water, which is 9 feet deep, from the cistern if the discharge of the pump is 2 feet above the top of the cistern.

3. Water stands 8 feet deep in a hemispherical reservoir 20 feet in diameter. Find the work done in pumping the water from the reservoir if the discharge of the pump is 4 feet above the level of the top of the reservoir.

4. Find the work done by a gas in expanding in accordance with the law $pv^{1.4} = C$ from a volume of 8 cubic feet to one of 10 cubic feet, if when $v = 7$ cubic feet, $p = 80$ pounds per square inch.

5. Find the work done in stretching a spring whose original length was 18 inches from a length of 20 inches to a length of 22 inches if a force of 80 pounds is required to stretch it to a length of 19 inches.

6. Find the work done in compressing a spring of original length 6 inches to a length of $4\frac{1}{2}$ inches, if a force of 300 pounds is required to compress it to a length of 5 inches.

7. A chain, 100 feet long and weighing 10 pounds per foot, is attached to, and hangs from, a windlass. Find, by the use of a definite integral, the work done in drawing up the chain.

8. Water is pumped from a round cistern whose median section is a parabola with its vertex at the surface of the ground. The cistern is 20 feet deep and the diameter of the bottom is 16 feet. Find the work done in emptying the cistern if the water is 12 feet deep and if the discharge of the pump is 5 feet above the top of the cistern.

9. Same as Exercise 8, but let the median section be a semi-ellipse, one extremity of the major axis being at the top of the cistern and the minor axis being a diameter of the bottom of the cistern.

10. A round water tank whose median section is a parabola whose depth is 20 feet, and whose diameter at the top is 16 feet, is filled by a pump placed 40 feet below the lowest point of the tank. Find the work done by the pump if the delivery pipe enters the tank at its lowest point. Express result in foot-tons.

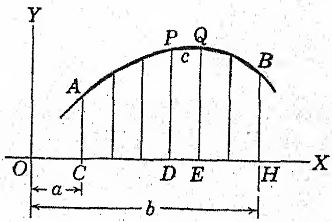

FIG. 43.

70. Length of Arc, Rectangular Coordinates. In §64 the length of arc of a curve was found by integrating its differential. We shall now express the length of arc by means of a definite integral.

To find the length of arc $APQB$, Fig. 43, divide CH into n equal parts of length Δx each. At the points of division erect ordinates dividing the arc AB into n parts of which PQ is one. The length of arc AB is defined by

$$s = \lim_{n \to \infty} \sum c,$$

where c is the length of a chord such as PQ. Then

$$s = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \sqrt{(\Delta x)^2 + (\Delta y)^2} = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x.$$

It follows from Duhamel's theorem that

$$s = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\Delta x = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx, \qquad (1)$$

since

$$\lim_{\Delta x \to 0} \frac{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = 1.$$

We can also write

$$s = \int_{x=a}^{x=b} ds,$$

where

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{(dx)^2 + (dy)^2}.$$

If it is more convenient to integrate with respect to y, ds may be written

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

The limits of integration will be the values of y which correspond to $x = a$ and $x = b$.

Exercises

1. Find the length of the curve $y = x^{\frac{3}{2}}$ between the points (0, 0) and (4, 8).

2. Find the length of $y^2 = 4x^3$ between the points (0, 0) and (10, $20\sqrt{10}$).

3. Find the length of $x^2 = 9y^3$ between the points (0, 0) and (3, 1).

4. Find the length of $(x + 2)^2 = 4y^3$ between the points in the first quadrant whose ordinates are 11 and 32.

5. Find the entire length of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

6. Find the length of $y = 6(x - 1)^{\frac{3}{2}}$ between the points (1, 0) and (5, 48).

7. Find the length of $x^2 = 9(y + 3)^3$ between the points (0, -3) and (81, 6).

8. Find, by calculus, the length of $y = mx + b$ intercepted by the lines $x = c$ and $x = d$.

9. Set up a definite integral in terms of x for the length of one quadrant of a circle of radius a.

10. Set up a definite integral in terms of x for the length of the upper half of the loop of $y^2 = x(x - 1)^2$.

71. Area of a Surface of Revolution. The portion MN, Fig. 40, of the curve $y = f(x)$, between the ordinates $x = a$ and $x = b$, is revolved about the X-axis. Find the area S of the surface generated.

Pass planes as in §68 perpendicular to the X-axis through the equidistant points of division of the interval $AB = b - a$. Denote the convex surface of the frustum of a cone generated by the revolution of $DEPQ$ by ΔF. The area S of the surface of revolution will be defined as the limit of the sum of the convex surfaces ΔF of these frustums as n becomes infinite, *i.e.*, as Δx approaches zero. Then

$$\begin{aligned} S &= \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \Delta F = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} 2\pi \frac{y + (y + \Delta y)}{2} c \\ &= \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} 2\pi \left(y + \frac{\Delta y}{2} \right) c \\ &= \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} 2\pi \left(y + \frac{\Delta y}{2} \right) \sqrt{1 + \left(\frac{\Delta y}{\Delta x} \right)^2} \Delta x. \end{aligned}$$

It follows from Duhamel's theorem that

$$S = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \Delta x = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx, \qquad (1)$$

since

$$\lim_{\Delta x \to 0} \frac{2\pi \left(y + \frac{\Delta y}{2} \right) \sqrt{1 + \left(\frac{\Delta y}{\Delta x} \right)^2} \Delta x}{2\pi y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \Delta x} = 1.$$

Equation (1) can be written

$$S = 2\pi \int_{x=a}^{x=b} y\, ds,$$

where ds is the differential of the length of arc. This form is easily remembered since it is the area of a circular strip of surface whose length is $2\pi y$ and whose width is ds.

If it is more convenient to integrate with respect to y, ds can be replaced by

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$

The limits are the values of y corresponding to $x = a$ and $x = b$. Thus

$$S = 2\pi \int_{y=y_1}^{y=y_2} y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = 2\pi \int_{y=y_1}^{y=y_2} y\,ds.$$

Exercises

1. The arc of the parabola $y^2 = x$ between $x = 0$ and $x = 4$ is revolved about the X-axis. Find the area of the surface generated.

2. The arc of the parabola $y^2 = x + 2$ which lies to the left of the Y-axis is revolved about the X-axis. Find the area of the surface generated.

3. Find the area of the surface generated by revolving about the Y-axis the part of the curve $x^2 = y + 4$ which lies below the X-axis.

4. Find the surface of the sphere generated by revolving $x^2 + y^2 = a^2$ about the X-axis.

5. Find the area of the surface generated by revolving $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the X-axis.

6. Find the area of the lateral surface of a right circular cone whose altitude is 12 feet and the radius of whose base is 4 feet.

7. Find the area of the surface generated by revolving about the X-axis the segment of the straight line connecting the points (0, 0) and (h, r).

8. Find the area of the surface generated by revolving about the Y-axis the part of the curve $x^2 = y + 9$ which lies below the X-axis.

72. Element of Integration. The first step in setting up a definite integral is to break up the area, volume, work, length, or whatever it is desired to calculate, into convenient parts which are infinitesimals as their number becomes infinite. These parts are then replaced by other infinitesimals of the typical

form $f(x)dx$, which must be so chosen that the limit of the ratio of each infinitesimal of the second set to the corresponding infinitesimal of the first set is one. $f(x)dx$ is called the "*element*" of the integral of the quantity which the integral represents. Thus the element of volume is $\pi y^2\,dx$, that of area is $y\,dx$, that of work is $F\,dx$.

If the magnitude which it is desired to calculate is broken up into suitable parts, the expressions for the elements can be written down at once. The best way of retaining in mind the formulas of §§**69, 70,** and **71** is to understand thoroughly how the elements are chosen. The process of writing down the element of integration at once becomes almost an intuitive one.

73. Water Pressure. The pressure at any given point in a liquid at rest is equal in all directions. The pressure at a given depth is equal to the force on a horizontal surface of unit area at that depth, *i.e.*, to the weight of the column of liquid supported by this surface. This weight is proportional to the depth. Hence the pressure at a depth h below the surface of the liquid is given by the formula $p = kh$. If the liquid is water and the depth h is measured in feet and the pressure in pounds per square foot, $k = 62.5$.

The method to be used in finding the force due to water pressure on any vertical surface is illustrated in the solution of the following problems:

Illustration 1. Find the force due to water pressure on one side of a gate in the shape of an isosceles triangle whose base is 6 feet and whose altitude is 5 feet, if it is immersed vertically in water with its vertex down and its base 4 feet below the surface of the water.

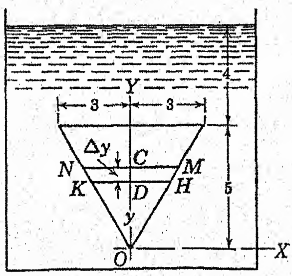

Fig. 44.

Choose axes as shown in Fig. 44. The altitude is supposed to be divided into n equal parts and through the points of division horizontal lines are supposed to be drawn dividing the surface into strips. The trapezoid $KHMN = \Delta A$ is a typical strip. Denote the

force on this strip by ΔP. The ordinate of this strip is y and its depth below the surface is $9 - y$. Hence the pressure on this strip is $k(9 - y)$. The area of the strip is approximately[1]

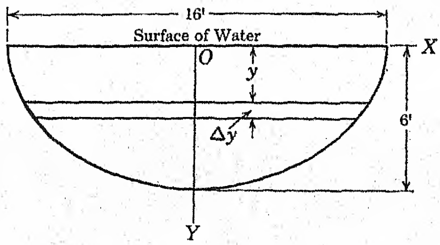

Fig. 45.

$2x\,\Delta y$ where x is the abscissa of the point H. Therefore the force on the strip is approximately

$$k(9 - y)(2x\,\Delta y)$$

and the total force on the gate is

$$P = \lim_{\Delta y \to 0} \sum_{y=0}^{y=5} 2k(9 - y)x\,\Delta y,$$

$$= 2k \int_0^5 (9 - y)x\,dy.$$

In this integral x can be expressed in terms of y by using the equation of the line OH, $y = \frac{5}{3}x$.
Then

$$P = \frac{6k}{5}\int_0^5 (9 - y)y\,dy$$

$$= \frac{6k}{5}\left[\frac{9y^2}{2} - \frac{y^3}{3}\right]_0^5 = 5312.5.$$

The force on one side of the gate is therefore 5312.5 pounds.

[1] In accordance with Duhamel's theorem the area of the strip ΔA can be replaced in

$$\lim_{\Delta y \to 0} \Sigma k(9 - y)\Delta A \text{ by } 2x\,\Delta y.$$

Illustration 2. Find the force due to water pressure on a vertical semi-elliptical gate, Fig. 45, whose major axis lies in the surface of the water, given that the semi-axes of the ellipse are 8 and 6 feet. Take the origin at the center, the axis of x horizontal, and the axis of y positive downward. As before, divide the surface of the gate into horizontal strips of width Δy. The area of a typical strip at depth y is approximately $2x\ \Delta y$ and the pressure at this depth is ky. The force on the strip in question is then approximately

$$2kyx\ \Delta y$$

and the total force on the gate is

$$P = \lim_{\Delta y \to 0} 2k \sum_{y=0}^{y=6} yx\ \Delta y = 2k \int_0^6 yx\ dy.$$

x is expressed in terms of y by means of the equation of the ellipse

$$\frac{x^2}{64} + \frac{y^2}{36} = 1.$$

Then

$$P = 2k\ \tfrac{4}{3} \int_0^6 y\sqrt{36 - y^2}\ dy.$$

The evaluation of the integral is left as an exercise.

Exercises

1. A cylindrical tank, 6 feet in diameter, lies with its axis horizontal. Find the force due to water pressure on one end of the tank if it is half full of water.

2. A watering trough, 12 feet long with ends 3 feet square, is full of water. Find the force due to water pressure on one end. On one side. On the bottom.

3. The section of a V-shaped trough is an equilateral triangle whose sides are 1 foot long. Find the force due to water pressure on one of the ends when the trough is filled with water.

4. Find the force due to water pressure on the vertical parabolic gate, Fig. 46: (*a*) if the edge AB lies in the surface of the water; (*b*) if the edge AB lies 4 feet below the surface.

5. Find the force due to water pressure on a vertical semicircular gate whose diameter, 8 feet long, lies in the surface of the water.

6. The upper edge of a vertical rectangular gate is 8 feet in length, horizontal, and 4 feet below the surface of the water. The vertical dimension of the gate is 6 feet. Find the force due to water pressure

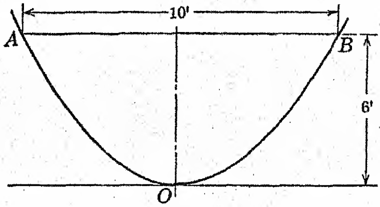

Fig. 46.

on the part of the gate which lies below a diagonal; above a diagonal.

7. Same as Exercise 4, but the gate is inverted, *i.e.*, the vertex O lies above the line AB: (*a*) if O lies in the surface of the water; (*b*) if O lies 4 feet below the surface.

8. A cylindrical tank, 8 feet in diameter, lies with its axis horizontal. Find the force due to water pressure on one end of the tank if the water is 4 feet deep.

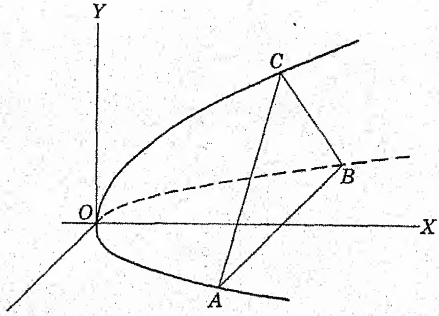

Fig. 47.

9. A watering trough whose vertical section is a parabola is 3 feet wide at the top and 3 feet deep. Find the force due to water pressure on one end of the trough if it is full of water.

10. Let the cross section of the trough of Exercise 9 be a semi-ellipse.

74. Additional Problems Involving the Use of the Definite Integral. In this section are given additional exercises in finding volumes of solids—solids for which parallel plane sections, in at least one direction, are similar figures.

Illustration 1. The cross section of a particular solid made by any plane perpendicular to the X-axis is an equilateral triangle. The mid-point of one side AB of the triangle, Fig. 47, lies on the X-axis and the opposite vertex C on the curve $y^2 = x$ in the XY-plane. Find the volume of the solid between the planes $x = 1$ and $x = 4$.

By following a line of reasoning similar to that of the preceding sections we see that the element of volume is $\frac{y^2\sqrt{3}}{3}dx$, and that the volume is

$$V = \frac{\sqrt{3}}{3}\int_1^4 y^2\,dx = \frac{\sqrt{3}}{3}\int_1^4 x\,dx = \frac{5\sqrt{3}}{2}.$$

Exercises

1. Same as *Illustration* 1 but with the center of the triangle on the X-axis.

2. A variable circle whose plane is perpendicular to, and whose center is on, the X-axis moves from left to right, from $x = 1$ to $x = 4$. Find the volume generated by the circle if its radius is proportional to x^2, and if its area is 9π when $x = 3$.

3. The axes of two right circular cylinders, each of radius a, intersect at right angles. Find the volume common to the two cylinders.

4. Find the volume between $x = 1$ and $x = 5$ of the solid for which the cross section made by a plane perpendicular to the X-axis and x units from the origin is a square of side $2x$.

5. Find the volume of one of the wedges cut from a circular log 12 inches in diameter by two planes, one perpendicular to the axis of the log, and the other inclined to it at an angle of 45°. The two planes intersect in a line meeting the axis of the cylinder at right angles.

Hint. In the plane perpendicular to the axis of the log, take the Y-axis along the edge of the wedge and the X-axis perpendicular to the edge and through the center of the circular section. Plane sections of the wedge perpendicular to the Y-axis are isosceles right triangles. Plane sections perpendicular to the X-axis are rectangles.

6. Find the volume of a regular triangular pyramid, each side of whose base is a and whose altitude is h.

7. Find the volume of a regular quadrangular pyramid, each side of whose base is a and whose altitude is h.

75. Arithmetic Mean. The arithmetic mean A of a series of n numbers, $a_1, a_2, a_3, \ldots, a_n$, is defined by the equation

$$nA = a_1 + a_2 + a_3 + \cdots + a_n,$$

or

$$\mathbf{A} = \frac{\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \cdots + \mathbf{a}_n}{\mathbf{n}}.$$

That is, A is such a number that if each number in the sum $a_1 + a_2 + a_3 + \cdots + a_n$ be replaced by it, this sum is unaltered.

76. Weighted Mean. In calculating the mean of a series of numbers, different weights are frequently assigned to the numbers according to their importance. For example, a student receives a grade of 95 in a five-credit course, a grade of 80 in a two-credit course, a grade of 86 in a three-credit course, and a grade of 90 in a four-credit course. The weighted mean of his grades is

$$M = \frac{95 \cdot 5 + 80 \cdot 2 + 86 \cdot 3 + 90 \cdot 4}{5 + 2 + 3 + 4} = 89 \cdot 5.$$

As a further example, the results of repeated measurements of the difference in elevation of two points are 16.32, 16.14, 16.08, 16.24, and 16.36 feet. To these measurements are assigned the weights 10, 3, 1, 8, and 6, respectively. These weights represent the relative importance to be attached to the respective measurements. The weighted mean of the observed differences in elevation is then

$$M = \frac{(16.32)(10) + (16.14)(3) + (16.08)(1) + (16.24)(8) + (16.36)(6)}{10 + 3 + 1 + 8 + 6} = 16.28.$$

In general, the weighted mean of the numbers $x_1, x_2, \ldots x_n$, whose weights are $w_1, w_2, \ldots w_n$, respectively, is given by

$$M = \frac{x_1w_1 + x_2w_2 + \cdots + x_nw_n}{w_1 + w_2 + \cdots + w_n} = \frac{\Sigma x_iw_i}{\Sigma w_i}.$$

A natural interpretation of this expression, suggesting the origin of the term, weighted mean, is obtained by considering a system of n masses resting on a horizontal beam AB, at distances $x_1, x_2, \ldots, x_n$ from the end A. At the points of the beam whose abscissas, measured from A, are $x_1, x_2, \ldots, x_n$, there act the vertical forces $w_1, w_2, \ldots, w_n$, equal to the weights of the respective masses. The weighted mean of the distances, $x_1, x_2, \ldots, x_n$, given by

$$\bar{x} = \frac{x_1w_1 + x_2w_2 + \cdots + x_nw_n}{w_1 + w_2 + \cdots + w_n},$$

is the distance from A of a point such that if the sum of the masses be concentrated at this point, the moment about A of the weight of the concentrated mass is equal to the sum of the moments about A of the weights of the masses as originally disturbed. For

$$\bar{x}(w_1 + w_2 + \cdots + w_n) = x_1w_1 + x_2w_2 + \cdots + x_nw_n.$$

77. Mean Value of a Function. The method of finding the mean of a series of numbers can be extended to find the mean value of a continuously varying magnitude. Consider the continuous function $y = f(x)$ between $x = a$ and $x = b$, represented in Fig. 48.

Subdivide the interval, $RS = b - a$, on the X-axis into n equal parts of length Δx by the points $x_0 = a, x_1, x_2, \cdots, x_{n-1}, x_n = b$. Let $y_0, y_1, \cdots, y_{n-1}$ be the values of y at the points of division $x_0 = a, x_1, \cdots, x_{n-1}$, respectively.

In the interval from x_0 to x_1, of length Δx, y has approximately the value y_0; in the next interval, also of length Δx, it has approximately the value y_1, and so on. Assign to each of these values, $y_0, y_1, \ldots, y_{n-1}$, a weight Δx equal to the length of the interval with which it is associated and calculate the weighted mean,

$$M_n = \frac{\Sigma y_i\,\Delta x}{\Sigma\,\Delta x}.$$

The numerator and the denominator can be divided by Δx, giving

$$M_n = \frac{\Sigma y_i}{n},$$

the arithmetic mean of the equally spaced ordinates, $y_0, y_1 \ldots,$

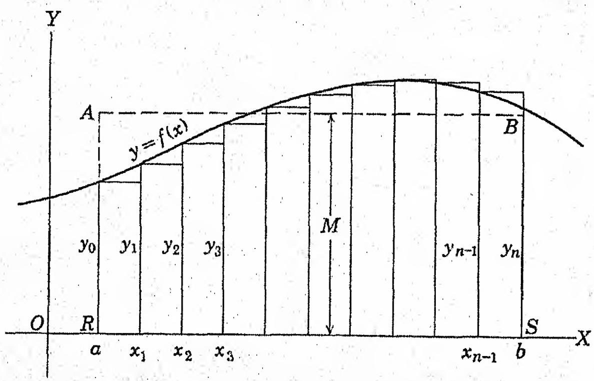

FIG. 48.

y_{n-1}. This result was to be expected since the weight, Δx, is the same for each of the n numbers.

The mean value M of $y = f(x)$ with respect to x between $x = a$ and $x = b$ will be defined as the limit which M_n approaches as the number of parts, n, into which the interval $b - a$ is divided, increases without limit. Then

$$M = \lim_{n \to \infty} M_n = \frac{\displaystyle\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} y_i\,\Delta x}{\displaystyle\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \Delta x} = \frac{\displaystyle\int_a^b y\,dx}{\displaystyle\int_a^b dx},$$

or

$$M = \frac{\int_a^b f(x)dx}{b-a}. \tag{1}$$

On writing equation (1) in the form

$$M(b-a) = \int_a^b f(x)dx \tag{2}$$

it is easily seen that M can be interpreted as the altitude of the rectangle, $RABS$, of base $b - a$, Fig. 48, whose area is equal to

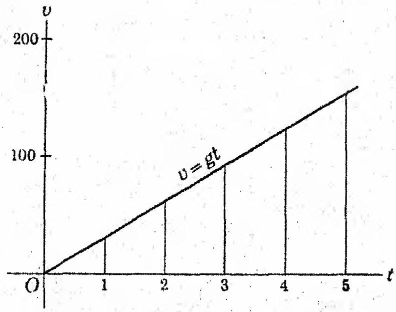

FIG. 49.

the area under the curve $y = f(x)$ between the limits $x = a$ and $x = b$.

Regarded as the altitude of a rectangle of base $b - a$ whose area is equal to that under the curve, M is called the *mean ordinate* of the curve $y = f(x)$ between the limits a and b.

In finding the mean value of a magnitude it is important to state the variable with respect to which the mean is determined. Thus if a body falls from rest from a height of 400 feet above the ground, the mean velocities with respect to the time and with respect to the distance are not the same.

If the velocity of the body is observed every second and the mean of these velocities taken, a certain mean value is found, a mean with respect to time. If, however, the velocity of the

body is observed every 50 feet during its descent, the mean of these velocities will be very different from the mean of the velocities observed every second. This fact will be evident from a consideration of Figs. 49 and 50.

In Fig. 49, v is plotted as a function of t, while in Fig. 50, it is plotted as a function of s. In drawing these figures and in the following discussion g is taken as 32 feet per second per second.

As the intervals at which the velocity is measured are diminished, the two mean values taken with respect to the time and with respect to the distance will clearly approach, as limiting values, the mean ordinates of the curves of Figs. 49 and 50, respectively, between the limits indicated.

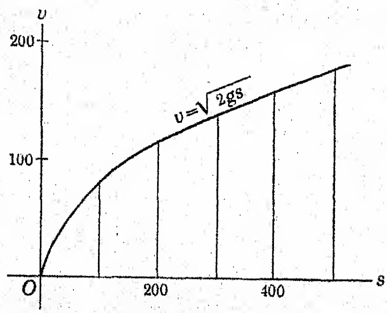

FIG. 50.

Thus by the method explained above, the mean of the velocity with respect to the time is given by

$$v = \frac{\int_0^5 gt\,dt}{5} = \frac{\frac{1}{2}gt^2\Big|_0^5}{5} = 80,$$

while the mean of the velocity with respect to the distance is given by

$$v = \frac{\int_0^{400} \sqrt{2gs}\,ds}{400} = \frac{\frac{2}{3}\sqrt{2g}\,s^{\frac{3}{2}}\Big|_0^{400}}{400} = 106.7.$$

In general, for a body falling from rest from a height s_0 in time t_0, the mean velocity with respect to the time is

$$v = \frac{\int_0^{t_0} gt\,dt}{t_0} = \frac{\frac{1}{2}gt^2\Big|_0^{t_0}}{t_0} = \tfrac{1}{2}gt_0 = \tfrac{1}{2}v_0,$$

where v_0 is the final velocity (see Fig. 49). The mean velocity with respect to the distance is

$$v = \frac{\int_0^{s_0} \sqrt{2gs}\,ds}{s_0} = \frac{\frac{2}{3}\sqrt{2g}\,s^{\frac{3}{2}}\Big|_0^{s_0}}{s_0} = \tfrac{2}{3}\sqrt{2gs_0} = \tfrac{2}{3}v_0,$$

where v_0 is again the final velocity.

As a further illustration of the importance of stating the variable with respect to which the mean is taken, consider the following simple problem.

Two automobiles make the round trip from Madison to Milwaukee, the distance between these cities being 80 miles. One automobile travels from Madison to Milwaukee at a speed of 20 miles per hour and returns at a speed of 30 miles per hour. The other makes the round trip at a speed of 25 miles per hour. Do the two automobiles make the round trip in the same time?

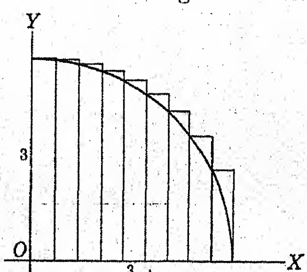

FIG. 51.

The first automobile takes the longer time. It requires $6\frac{2}{3}$ hours, while the second requires 6.4 hours.

The average speed of the first, taken with respect to the distance, is 25 miles per hour, the same as that of the second. But the average speed of the first taken with respect to the time, obtained by dividing 160 by $6\frac{2}{3}$, is 24 miles per hour. Its average speed with respect to the time is slower than that of the second automobile.

Illustration 1. Find the mean value of $y = \sqrt{9 - x^2}$ with respect to x between $x = 0$ and $x = 3$ (see Fig. 51).

$$M = \frac{\int_0^3 y\,dx}{3} = \frac{\int_0^3 \sqrt{9 - x^2}\,dx}{3}.$$

The numerator represents the area of one quadrant of the circle $x^2 + y^2 = 9$. Hence it is equal to $\frac{9\pi}{4}$. Hence

$$M = \frac{\frac{9\pi}{4}}{3} = \frac{3\pi}{4} = 2.36.$$

This is the mean ordinate of the portion of the circle $x^2 + y^2 = 9$ which lies in the first quadrant.

Illustration 2. Find the mean value between the points for which $x = 0$ and $x = 3$, of $y = \sqrt{9 - x^2}$ with respect to s, the length of arc of the curve measured from the point (0, 3) (Fig. 52).

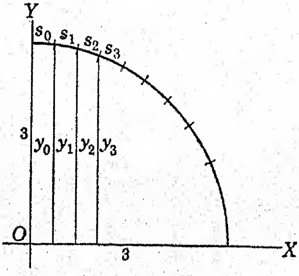

Fig. 52.

Subdivide the interval on the X-axis between $x = 0$ and $x = 3$ into n equal parts of length Δx by the points $x_0 = 0$, x_1, x_2, . . . , x_{n-1}, $x_n = 3$. Ordinates erected at these points subdivide the arc into portions of length Δs_0, Δs_1, . . . , Δs_{n-1}.

For points in the first portion of arc of length Δs_0, y has approximately the value y_0; for points in the next portion of length Δs_1, it has approximately the value y_1, etc. Assign to these values of y the weights Δs_0, Δs_1, . . . , Δs_{n-1}, corresponding to the lengths of the portion of arc with which they are associated. The weighted mean of these values is

$$M_n = \frac{\sum_{x=0}^{x=3} y_i\,\Delta s_i}{\sum_{x=0}^{x=3} \Delta s_i}.$$

Then

$$M = \lim_{n \to \infty} M_n = \frac{\lim\limits_{\Delta x \to 0} \sum\limits_{x=0}^{x=3} y_i \, \Delta s_i}{\lim\limits_{\Delta x \to 0} \sum\limits_{x=0}^{x=3} \Delta s_i} = \frac{\lim\limits_{\Delta x \to 0} \sum\limits_{x=0}^{x=3} y_i \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Delta x}{\lim\limits_{\Delta x \to 0} \sum\limits_{x=0}^{x=3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Delta x}$$

$$= \frac{\int_0^3 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx}{\int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx}.$$

The denominator is equal to $\frac{3\pi}{2}$, one-fourth of the circumference of the circle. The numerator is

$$\int_0^3 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^3 y \sqrt{1 + \frac{x^2}{y^2}} \, dx = \int_0^3 \sqrt{x^2 + y^2} \, dx$$

$$= 3 \int_0^3 dx = 9.$$

Hence

$$M = \frac{9}{\frac{3\pi}{2}} = \frac{6}{\pi} = 1.91.$$

An interpretation of this result can be found by considering the arc of the quadrant of the circle to be a fine wire of uniform cross section and density, lying in the horizontal XY-plane. M is the distance from the X-axis at which the mass of the wire must be concentrated in order that the moment about the X-axis of the force of gravity acting upon the concentrated mass shall be equal to the moment of the forces of gravity acting upon the elements of mass of the given wire.

Illustration 3. Find the mean distance from the Y-axis of the area in the first quadrant enclosed by the curve $y = \sqrt{9 - x^2}$ and the axes of coordinates.

The mean of x is to be found with respect to the area. Divide the area into n vertical strips $\Delta A_0, \Delta A_1, \ldots, \Delta A_{n-1}$ with equal bases of length Δx on the X-axis (see Fig. 51). The area of a typical strip ΔA_i is approximately $y_i\,\Delta x$. The distance x_i of any strip from the Y-axis is approximately the same for all points of the strip. Assign to each x_i the weight $y_i\Delta x$ and calculate the weighted mean of these values of x.

$$M_n = \frac{\sum_{x=0}^{x=3} x_i y_i\,\Delta x}{\sum_{x=0}^{x=3} y_i\,\Delta x}.$$

Then

$$M = \lim_{n\to\infty} M_n = \frac{\lim_{\Delta x\to 0}\sum_{x=0}^{x=3} x_i y_i\,\Delta x}{\lim_{\Delta x\to 0}\sum_{x=0}^{x=3} y_i\,\Delta x},$$

or

$$M = \frac{\int_0^3 xy\,dx}{\int_0^3 y\,dx} = \frac{\int_0^3 x\sqrt{9-x^2}\,dx}{\int_0^3 \sqrt{9-x^2}\,dx} = \frac{-\frac{1}{3}(9-x^2)^{\frac{3}{2}}\Big|_0^3}{\frac{9\pi}{4}}$$

$$= \frac{9}{\frac{9\pi}{4}} = \frac{4}{\pi} = 1.27.$$

An interesting interpretation of this result can be found by considering the area in question to be the surface of a uniform thin plate lying in the horizontal XY-plane. M is the distance from the Y-axis at which the mass of the plate must be concentrated in order that the moment about the Y-axis of the force of gravity acting upon the concentrated mass shall be the same as the

moment about the Y-axis of the forces of gravity acting upon the elements of mass of the given plate.

Exercises

1. Find the mean ordinate of $y = \sqrt{x}$ between $x = 0$ and $x = 3$.

2. Find the mean with respect to x of $y = x^2$ between $x = 1$ and $x = 4$.

3. A solid is generated by revolving about the X-axis that portion of $y = x + x^2$ which lies between $x = 2$ and $x = 5$. Find the mean with respect to x of the areas of the circular cross sections of the solid made by planes perpendicular to the axis of x. Interpret the result.

4. A solid is generated by revolving about the X-axis the portion of $y = x^2 + 1$ which lies between $x = 1$ and $x = 5$. Find the mean with respect to x of the areas of the circular cross sections of the solid made by planes perpendicular to the axis of x.

5. A body falls from rest from a height of 1600 feet above the ground. Find the mean velocity; (a) with respect to the time; (b) with respect to the distance fallen.

6. Find the mean distance from the Y-axis of the area enclosed by $y = x$, the X-axis, and $x = 2$.

7. Find the mean distance from the Y-axis of the portion of the arc of the circle $x^2 + y^2 = 9$ which lies in the first quadrant.

8. The altitude of a right circular cone is 10 inches and the radius of its base is 4 inches. The density in any plane perpendicular to the axis of the cone is equal to kx, where x is the distance of the plane from the vertex. Find the mean of the density with respect to the volume.

9. Find the mean distance from the Y-axis of the area in the first quadrant bounded by $b^2x^2 + a^2y^2 = a^2b^2$.

10. Find the mean distance of the volume of a hemisphere from its base.

11. Find the mean distance of the surface of a hemisphere from its base.

12. Find the mean distance from the Y-axis of the area of the triangle enclosed by $2y = x$, the X-axis, and the line $x = 2$.

13. Find the mean of the square of the ordinate of $y = x^2$ between $x = 1$ and $x = 2$. The square root of this mean is called the root mean square of y between $x = 1$ and $x = 2$.

14. Find the root mean square of $y = \sqrt{9 - x^2}$ between $x = 0$ and $x = 3$.

CHAPTER VIII

CIRCULAR FUNCTIONS. INVERSE CIRCULAR FUNCTIONS

Up to this point only functions have been discussed which are simple algebraic combinations of powers of the dependent variable. Many interesting applications of the calculus to the study of these functions have been given. We shall now take up the study of the application of the methods of the calculus to another very important class of functions, the circular functions.

78. Derivative of sin u. Let

$$\begin{aligned} y &= \sin u. \\ y + \Delta y &= \sin (u + \Delta u), \\ \Delta y &= \sin (u + \Delta u) - \sin u \\ &= \sin u \cos \Delta u + \cos u \sin \Delta u - \sin u, \\ \frac{\Delta y}{\Delta u} &= \frac{\cos u \sin \Delta u}{\Delta u} - \frac{\sin u(1 - \cos \Delta u)}{\Delta u}. \end{aligned}$$

By §55

$$\lim_{\Delta u \to 0} \frac{\sin \Delta u}{\Delta u} = 1,$$

and by §57

$$\lim_{\Delta u \to 0} \frac{1 - \cos \Delta u}{\Delta u} = 0.$$

Hence

$$\frac{dy}{du} = \cos u. \tag{1}$$

If u is a function of x,

$$\frac{dy}{dx} = \cos u \frac{du}{dx}. \tag{2}$$

The corresponding formula for dy is

$$dy = \cos u\, du. \tag{3}$$

It has thus been shown that

$$\frac{\mathbf{d(sin\ u)}}{\mathbf{dx}} = \mathbf{cos\ u}\ \frac{\mathbf{du}}{\mathbf{dx}} \tag{4}$$

and

$$\mathbf{d(sin\ u) = cos\ u\ du.}$$

Well-known properties of the function $y = \sin x$ can be verified by formula (1). Since $\cos x$ is positive when $0 < x < \frac{\pi}{2}$ and when $\frac{3\pi}{2} < x < 2\pi$, $\sin x$ is an increasing function within these limits. In like manner $\sin x$ is a decreasing function when $\frac{\pi}{2} < x < \frac{3\pi}{2}$, for between these limits $\cos x$ is negative. Thus from (1) we see that $\sin x$ has a maximum value when $x = \frac{\pi}{2} \pm 2n\pi$, and a minimum value when $x = \frac{3\pi}{2} \pm 2n\pi$, where n is a positive integer.

It is to be observed that in the derivation of the formula for the derivative of $\sin u$, the angle u was considered to be measured in radians. This is evident since use was made of the result of §55.

79. Derivatives of cos u, tan u, cot u, sec u, csc u. The derivatives of the remaining circular functions can be obtained from that of the sine.

Let $y = \cos u$. Then

$$y = \sin\left(\frac{\pi}{2} - u\right)$$

and

$$\begin{aligned}\frac{dy}{dx} &= \cos\left(\frac{\pi}{2} - u\right)\frac{d\left(\frac{\pi}{2} - u\right)}{dx}\\ &= \cos\left(\frac{\pi}{2} - u\right)\left(-\frac{du}{dx}\right)\\ &= -\sin u\,\frac{du}{dx}.\end{aligned}$$

Hence

$$\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx} \tag{1}$$

and

$$d(\cos u) = -\sin u\, du.$$

By writing

$$\tan u = \frac{\sin u}{\cos u},$$

$$\cot u = \frac{\cos u}{\sin u},$$

$$\sec u = \frac{1}{\cos u},$$

and

$$\csc u = \frac{1}{\sin u},$$

the student will show that

$$\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}, \quad \text{or } d(\tan u) = \sec^2 u\, du, \tag{2}$$

$$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}, \quad \text{or } d(\cot u) = -\csc^2 u\, du, \tag{3}$$

$$\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx}, \quad \text{or } d(\sec u) = \sec u \tan u\, du, \tag{4}$$

$$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx}, \quad \text{or } d(\csc u) = -\csc u \cot u\, du. \tag{5}$$

Illustration 1. Find the first and second derivatives of $3 \sin (2x - 5)$.

$$\frac{d[3 \sin (2x - 5)]}{dx} = 3\frac{d[\sin (2x - 5)]}{dx}$$

$$= 3 \cos (2x - 5)\frac{d(2x - 5)}{dx}$$

$$= 6 \cos (2x - 5).$$

Differentiating again,

$$\frac{d^2[3 \sin (2x - 5)]}{dx^2} = 6\frac{d[\cos (2x - 5)]}{dx}$$
$$= -6 \sin (2x - 5)\frac{d(2x - 5)}{dx}$$
$$= -12 \sin (2x - 5).$$

Illustration 2. If $y = \sin 2x \cos x$, find $\frac{dy}{dx}$. Since $\sin 2x \cos x$ is the product of two functions, apply formula (1), §**39**.

$$\frac{dy}{dx} = \sin 2x(- \sin x)\frac{dx}{dx} + (\cos x)(\cos 2x)\frac{d2x}{dx}$$
$$= 2 \cos x \cos 2x - \sin x \sin 2x.$$

Illustration 3. If $y = 3 \sin x + 4 \cos x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = 3 \cos x - 4 \sin x$$
$$\frac{d^2y}{dx^2} = -(3 \sin x + 4 \cos x) = -y.$$

Illustration 4. If $y = \tan^3 3x = (\tan 3x)^3$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. The function is of the form $y = u^n$. Hence

$$\frac{dy}{dx} = 3(\tan 3x)^2 \frac{d(\tan 3x)}{dx}$$
$$= 3 \tan^2 3x \sec^2 3x \frac{d3x}{dx}$$
$$= 9 \tan^2 3x \sec^2 3x.$$
$$\frac{d^2y}{dx^2} = 9\left[\tan^2 3x \frac{d(\sec^2 3x)}{dx} + \sec^2 3x \frac{d(\tan^2 3x)}{dx}\right]$$
$$= 54(\tan^3 3x \sec^2 3x + \tan 3x \sec^4 3x)$$
$$= 54 \tan 3x \sec^2 3x(\tan^2 3x + \sec^2 3x).$$

Exercises

In Exercises 1 to 10, verify the differentiation.

1. $y = \sin 5x,\qquad \dfrac{dy}{dx} = 5 \cos 5x,\qquad \dfrac{d^2y}{dx^2} = -25y.$

2. $y = \cos 3x,\qquad \dfrac{dy}{dx} = -3 \sin 3x,\qquad \dfrac{d^2y}{dx^2} = -9y.$

3. $y = \tan 2x,\qquad \dfrac{dy}{dx} = 2 \sec^2 2x,$

$$\frac{d^2y}{dx^2} = 8 \sec^2 2x \tan 2x.$$

4. $y = \sin x \cos 2x,\qquad \dfrac{dy}{dx} = \cos 2x \cos x - 2 \sin 2x \sin x$

5. $y = \sin \dfrac{3x - 2}{5}.\qquad \dfrac{dy}{dx} = \frac{3}{5} \cos \dfrac{3x - 2}{5},$

$$\frac{d^2y}{dx^2} = -\tfrac{9}{25}y.$$

6. $y = \tan^3 5x,\qquad dy = 15 \tan^2 5x \sec^2 5x\, dx.$

7. $y = \sec^4 3x,\qquad dy = 12 \sec^4 3x \tan 3x\, dx.$

8. $\begin{cases} y = a(1 - \cos \theta), & dy = a \sin \theta\, d\theta. \\ x = a(\theta - \sin \theta), & dx = a(1 - \cos \theta)d\theta. \end{cases}$

9. $y = a \sin \left(\dfrac{2\pi t}{T} - e\right),\qquad \dfrac{dy}{dt} = \dfrac{2a\pi}{T} \cos \left(\dfrac{2\pi t}{T} - e\right).$

10. $y = x \sin x,\qquad dy = (x \cos x + \sin x)\, dx.$

11. From the results of Exercise 8, show that $\dfrac{dy}{dx} = \cot \dfrac{\theta}{2}.$

Find dy in Exercises 12 to 31.

12. $y = \sin^3 2x.$
13. $y = \tan^2 5x.$
14. $y = \cos^3 4x.$
15. $y = \sin^2 (3x - 1).$
16. $y = \cos^3 (2x - 3).$
17. $y = \sec^3 2x.$
18. $y = \sin^3 (3 - 2x).$
19. $y = x \cos x.$
20. $y = \sqrt{\sin 3x}.$
21. $y = \cos (x^2 - 3x + 2).$
22. $y = \sin x^2.$
23. $y = x \sin 2x.$
24. $y = \tan^3 (4x + 2).$
25. $y = \cot^2 (2x - 5).$
26. $y = \csc^3 (3x - 4).$
27. $y = \tan^4 (5 - 3x).$
28. $y = x \tan x.$
29. $y = \cos^3 (1 - x^2).$
30. $y = \cot^4 3x.$
31. $y = \sec^4 2x.$
32. Find $\dfrac{dy}{dx}$ if $x = \sqrt{1 - \sin y}.$

33. An airplane 1 mile high is flying horizontally with a velocity of 100 miles per hour, directly away from an observer. At what rate is the angle of elevation of the airplane changing when the point directly under the airplane is 1000 feet from the observer?

34. Given that $x = 10 \cos \theta$ and that θ is increasing at the rate of 0.5 radian per second, find the rate at which x is changing when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, 0, \frac{\pi}{2}, \pi$. Interpret x as the abscissa of a point on the circumference of the circle $x^2 + y^2 = 100$.

35. A searchlight on a ship 3 miles from a straight shore is being turned about a vertical axis at the rate of 0.2 radian per second. At what rate is the spot of light moving along the shore when the beam of light makes an angle of 45° with the shore?

36. A man walks at the constant rate of 6 feet per second along the diameter of a semicircular courtyard whose radius is 40 feet. The sun's rays are perpendicular to the diameter. How fast is the man's shadow moving along the semicircular wall of the courtyard when he is 25 feet from the end of the diameter?

37. For what values of x is $y = 5 \sin x + 12 \cos x$ a maximum or a minimum? Answer by using calculus and also by writing the equation in the form, $y = c \cos (x - \alpha)$.

38. For what values of x is $y = 2 \cos x - \cos 2x$ a maximum or a minimum? Sketch the curve by sketching $y = 2 \cos x$ and $y = \cos 2x$.

39. For what values of x is $y = \sin 2x + 2 \sin x$ a maximum or a minimum? Sketch the curve as in Exercise 38.

40. A drawbridge 30 feet long is raised by chains attached to the end of the bridge and passing over a pulley 30 feet above the hinge of the bridge. The chain is being drawn in at the rate of 5 feet a minute. Horizontal rays of light fall on the bridge and it casts a shadow on a vertical wall. How fast is the shadow moving up the wall when 18 feet of the chain have been drawn in?

41. The sides of a V-shaped trough are 10 inches wide. What should be the angle between the sides for a trough of maximum capacity?

42. Find the length of the shortest beam that can be used to brace a wall, if the beam is to pass over a second wall 8 feet high and 10 feet from the first.

43. A sector is cut from a circular piece of tin. The cut edges of the remaining portion of the sheet are then brought together to form a cone. Find the angle of the sector to be cut out in order that the volume of the cone shall be as great as possible.

44. The horizontal component of the tension in the guy wire BC, Fig. 53, is to balance the horizontal pull P. If the strength of the wire varies as its cross section, and if its cost varies as its weight, find the angle θ such that the cost of the guy wire shall be a minimum.

45. A steel girder 30 feet long is moved on rollers along a passageway 10 feet wide, and through the door AB, Fig. 54, at the end of the passageway. Neglecting the width of the girder, how wide must the door be in order to allow the girder to pass through?

46. A sign 10 feet high is fastened to the side of a building so that the lower edge is 25 feet from the ground. How far from the building should an observer on the ground stand in order that he may see the sign to the best advantage, *i.e.*, in order that the angle at his eye subtended by the sign may be the greatest possible? The observer's eye is $5\frac{1}{2}$ feet from the ground.

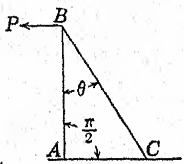

FIG. 53.

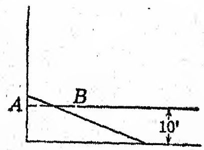

FIG. 54.

47. It is desired to make a gutter, whose cross section shall be a segment of a circle, by bending a strip of tin of width a. Find the radius of the cross section of maximum carrying capacity.

48. Find the dimensions of the largest rectangle that can be inscribed in the ellipse $x = 4 \cos \theta$, $y = 3 \sin \theta$.

49. Given $\sin 30° = 0.5$. Find the approximate value of $\sin 31°$ to four decimal places.

HINT. $y = \sin x$. Find dy when $x = \frac{\pi}{6}$ and $dx = \frac{\pi}{180} = 0.0174$.

50. The height of a vertical cliff is to be found by measuring the angle of elevation of its top and the perpendicular distance of its base from the observer. If the distance to the base is found with sufficient accuracy to be 512.2 feet and the angle of elevation is found to be 42° with a possible error of 0.5°, what is, approximately, the possible error in the height of the cliff as computed from these measurements?

51. If $x = 20 \cos \theta$, find the approximate change in x when θ changes from 60° to 62°.

52. Find the maximum and minimum points and the points of inflection of the curve $y = \sin 2x + x$. Sketch the curve.

53. Find the maximum and minimum points of the curve $y = \sin^2 x + \cos x$.

80. Integration of Circular Functions.

Illustration 1. If $\dfrac{dy}{dx} = \cos x$, find y.

$$\frac{dy}{dx} = \cos x.$$

Hence

$$y = \sin x + C.$$

Illustration 2. If $\dfrac{dy}{dx} = \sin 3x$, find y.

$$\frac{dy}{dx} = -\tfrac{1}{3}\left[-\sin 3x\frac{d\,3x}{dx}\right].$$

Hence

$$y = -\tfrac{1}{3}\cos 3x + C.$$

Illustration 3. If $\dfrac{dy}{dx} = \sec^2 2x$, find y.

$$\frac{dy}{dx} = \tfrac{1}{2}\left[\sec^2 2x\,\frac{d\,2x}{dx}\right].$$

Hence

$$y = \tfrac{1}{2}\tan 2x + C.$$

Illustration 4. If $\dfrac{dy}{dx} = \sec 5x \tan 5x$, find y.

$$\frac{dy}{dx} = \tfrac{1}{5}\left[\sec 5x \tan 5x\,\frac{d\,5x}{dx}\right].$$

Hence

$$y = \tfrac{1}{5}\sec 5x + C.$$

Illustration 5. If $dy = \sin^2 2x \cos 2x\, dx$, find y.

$$y = \int (\sin 2x)^2 \cos 2x\, dx$$

$$= \tfrac{1}{3}\tfrac{1}{2}\int 3(\sin 2x)^2 \cos 2x\, d(2x)$$

$$= \tfrac{1}{6}\int 3(\sin 2x)^2 d(\sin 2x).$$

Hence

$$y = \tfrac{1}{6}(\sin 2x)^3 + C.$$

Illustration 6. If $dy = \tan^3 5x \sec^2 5x\, dx$, find y.

$$y = \int \tan^3 5x \sec^2 5x\, dx$$

$$= \tfrac{1}{4}\tfrac{1}{5}\int 4(\tan 5x)^3 \sec^2 5x\, d(5x)$$

$$= \tfrac{1}{20}\int 4(\tan 5x)^3\, d(\tan 5x).$$

Hence

$$y = \tfrac{1}{20}(\tan 5x)^4 + C.$$

In the next four illustrations use will be made of the following relations:

$$\begin{aligned}
\sin(a+b) + \sin(a-b) &= 2\sin a \cos b.\\
\sin(a+b) - \sin(a-b) &= 2\cos a \sin b.\\
\cos(a+b) + \cos(a-b) &= 2\cos a \cos b.\\
\cos(a+b) - \cos(a-b) &= -2\sin a \sin b.
\end{aligned}$$

Illustration 7.

$$\int \sin 5x \cos 3x\, dx = \int \tfrac{1}{2}[\sin(5x+3x) + \sin(5x-3x)]dx$$

$$= \tfrac{1}{2}\int \sin 8x\, dx + \tfrac{1}{2}\int \sin 2x\, dx$$

$$= -\tfrac{1}{16}\cos 8x - \tfrac{1}{4}\cos 2x + C.$$

Illustration 8.

$$\int \cos 7x \sin 3x\,dx = \int \tfrac{1}{2}[\sin(3x + 7x) + \sin(3x - 7x)]dx$$
$$= \tfrac{1}{2}\int \sin 10x\,dx - \tfrac{1}{2}\int \sin 4x\,dx$$
$$= -\tfrac{1}{20}\cos 10x + \tfrac{1}{8}\cos 4x + C.$$

Illustration 9.

$$\int \cos 4x \cos 7x\,dx = \int \tfrac{1}{2}[\cos(7x + 4x) + \cos(7x - 4x)]dx$$
$$= \tfrac{1}{2}\int \cos 11x\,dx + \tfrac{1}{2}\int \cos 3x\,dx$$
$$= \tfrac{1}{22}\sin 11x + \tfrac{1}{6}\sin 3x + C.$$

Illustration 10.

$$\int \sin 4x \sin 2x\,dx = -\tfrac{1}{2}\int [\cos(4x + 2x) - \cos(4x - 2x)]dx$$
$$= -\tfrac{1}{2}\int \cos 6x\,dx + \tfrac{1}{2}\int \cos 2x\,dx$$
$$= -\tfrac{1}{12}\sin 6x + \tfrac{1}{4}\sin 2x + C.$$

To integrate $\sin^2 x\,dx$ and $\cos^2 x\,dx$, recall the relations:

$$\left(\begin{aligned} \sin^2 x &= \frac{1 - \cos 2x}{2} \\ \cos^2 x &= \frac{1 + \cos 2x}{2}. \end{aligned}\right)$$

Illustration 11.

$$\int \sin^2 3x\,dx = \int \frac{1 - \cos 6x}{2}\,dx$$
$$= \frac{x}{2} - \frac{1}{12}\sin 6x + C.$$

Exercises

Integrate:

1. $dy = \cos 3x\,dx.$
2. $dy = \sin 2x\,dx.$
3. $dy = \sin x \cos x\,dx.$
4. $dy = \sec^2 5x\,dx.$
5. $dy = \tan^2 5x \sec^2 5x\,dx.$
6. $dy = \sec 3x \tan 3x\,dx.$
7. $dy = \sec^4 x \tan x\,dx$
   $= \sec^3 x \sec x \tan x\,dx.$
8. $dy = \cos^4 x \sin x\,dx.$
9. $dy = \sin^2 4x \cos 4x\,dx.$
10. $dy = \tan 3x \sec^2 3x\,dx.$
11. $dy = \cos^3 2x \sin 2x\,dx.$
12. $\int \sqrt{\sin 3x} \cos 3x\,dx.$
13. $\int \sin^4 (2x - 3) \cos (2x - 3) dx.$
14. $\int \tan^3 (3 - 2x) \sec^2 (3 - 2x) dx.$
15. $\int \sin^2 x\,dx.$
16. $\int \cos^2 x\,dx.$
17. $\int \cos^2 3x\,dx.$
18. $\int \sin^2 5x\,dx.$
19. $\int \sin^2 4x\,dx.$
20. $\int \cos^2 4x\,dx.$
21. $\int \sec^3 2x \tan 2x\,dx.$
22. $\int \cos^2 (2x - 5) \sin (2x - 5) dx.$
23. $\int \sqrt{\cos 2x} \sin 2x\,dx.$
24. $\int \sin 6x \cos 2x\,dx.$
25. $\int \cos 4x \cos 3x\,dx.$
26. $\int \cos 5x \sin 2x\,dx.$
27. $\int \sin 8x \sin 3x\,dx.$
28. $\int \sin 4x \cos 7x\,dx.$
29. $\int \cos 5x \cos 9x\,dx.$
30. $\int \sin \omega t \cos at\,dt.$

31. $\int \cos \omega t \cos at\, dt.$

32. Find the area under one arch of the sine curve.

33. Find the area under one arch of the curve $y = 2a^2 \sin^2 x$.

34. Find the area under one arch of the curve $y = \cos x - \sin x$.

35. Find the area under one arch of the curve $y = \cos x - \sqrt{3} \sin x$.

36. The slope of a curve at any point is equal to $\sin x$. Find the equation of the curve if it passes through $\left(\frac{\pi}{2}, 1\right)$.

37. A point moves on a straight line with an acceleration equal to $\sin t$. Find an expression for the displacement s in terms of t, if $s = 0$ when $t = 0$, and if $s = \pi$ when $t = \pi$.

38. The slope of a curve at any point is equal to $\cos x - 3 \sin x$. Find the equation of the curve if it passes through the point $(0, 0)$.

39. Find the equation of the curve passing through the point $(0, 0)$ whose subnormal at any point (x, y) is equal to $\sin x$.

40. Find the equation of the curve passing through the point $\left(\frac{\pi}{4}, 0\right)$ whose subtangent at any point (x, y) is equal to $y \cos^2 x$.

41. Find the volume bounded by the surface obtained by revolving one arch of $y = \sin x$ about the X-axis.

42. Find the mean ordinate of the curve $y = \sin x$ between $x = 0$ and $x = \pi$.

43. Find the mean with respect to x of the square of $y = \sin x$ between $x = 0$ and $x = \pi$.

44. A particle moves along the X-axis in accordance with the law $x = a \cos \omega t$, ω being a constant. Find the mean of the velocity with respect to t between $t = 0$ and $t = \frac{\pi}{\omega}$. Find the mean with respect to t of the square of the velocity between $t = 0$ and $t = \frac{2\pi}{\omega}$.

81. Parametric Equations Containing Circular Functions.

Illustration 1. Find the length of the portion of the circle $x = a \cos \theta$, $y = a \sin \theta$, which lies in the first quadrant.

In case the equations of a curve are given in parametric form, it is often advantageous to use the expression for ds in the form

$$ds = \sqrt{dx^2 + dy^2}.$$

Now

$$dx = -a \sin \theta \, d\theta$$
$$dy = a \cos \theta \, d\theta.$$

Then

$$\begin{aligned} ds &= \sqrt{a^2(\sin^2 \theta + \cos^2 \theta)} \, d\theta \\ &= a \, d\theta, \end{aligned}$$

and

$$s = a \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi a}{2}.$$

Illustration 2. Find the area in the first quadrant bounded by the circle $x = a \cos \theta$, $y = a \sin \theta$, and the axes of coordinates.

$$A = \int^b y \, dx.$$

Now $y = a \sin \theta$ and $dx = -a \sin \theta \, d\theta$. From the relation $x = a \cos \theta$, note that as x increases from 0 to a, θ changes from $\frac{\pi}{2}$ to 0. Hence

$$\begin{aligned} A &= -a^2 \int_{\frac{\pi}{2}}^{0} \sin^2 \theta \, d\theta = a^2 \int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta \\ &= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta \\ &= \frac{a^2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) \Big|_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4}. \end{aligned}$$

Exercises

1. Find the area of the ellipse whose parametric equations are $x = a \cos \alpha$, $y = b \sin \alpha$.

2. Find the area under one arch of the cycloid,

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta).$$

3. Find the length of the curve,

$$x = a\cos^3\phi$$
$$y = a\sin^3\phi.$$

4. Find the length of one arch of the cycloid,

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta).$$

5. Find the mean with respect to x, of the ordinates of the arc in the first quadrant, of the circle $x = a\cos\theta$, $y = a\sin\theta$. Find the mean with respect to θ of the ordinates of the same arc.

6. Find the mean distance from the Y-axis, of the area in the first quadrant, of the ellipse $x = a\cos\theta$, $y = b\sin\theta$.

7. Find the mean distance from the Y-axis, of the arc in the first quadrant, of the circle $x = a\cos\theta$, $y = a\sin\theta$.

8. Find an expression in terms of θ and $\dfrac{d\theta}{dt}$ for the speed of a particle moving on the cycloid

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta).$$

9. Find an expression in terms of θ and $\dfrac{d\theta}{dt}$ for the speed of a particle moving on the ellipse $x = a\cos\theta$, $y = b\sin\theta$.

82. The Inverse Circular Functions. The graphs of the inverse circular functions, $y = \sin^{-1} x$, $y = \cos^{-1} x$, and $y = \tan^{-1} x$, are shown in Figs. 55, 56, and 57, respectively. It is obvious that these functions are multiple-valued functions. The first two are defined only for values of x from -1 to $+1$ inclusive, and for any value of x within these limits each function has an unlimited number of values. The third function is defined for all values of x, and for any value of x this function also has an unlimited number of values. It will be sufficient for the purpose of the calculus and its applications to consider the one-valued functions obtained by adopting the following conventions: If $y = \sin^{-1} x$, y will be

taken to lie between the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, inclusive. If $y = \cos^{-1} x$, y will be taken to lie between the limits 0 and π, inclusive.

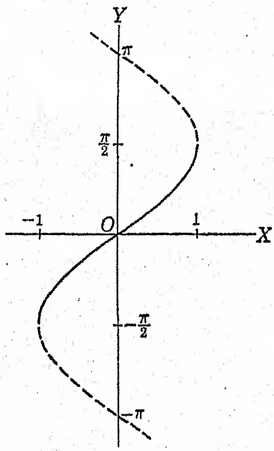

FIG. 55.

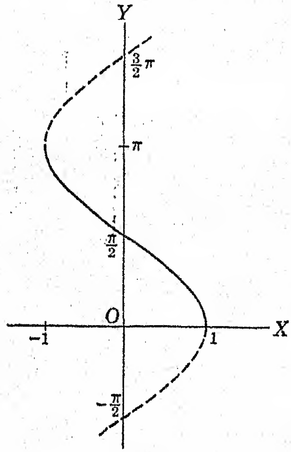

FIG. 56.

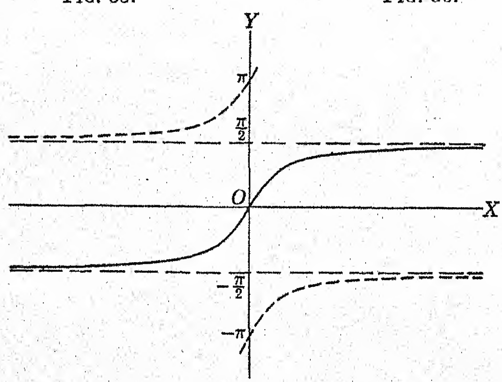

FIG. 57.

If $y = \tan^{-1} x$, y will be taken to lie between the limits $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, inclusive.

Formulas for the derivatives of the inverse circular functions can be readily obtained from those of §78 and §79.

Let $y = \sin^{-1} u$. Then $\sin y = u$. Differentiation gives

$$\cos y \frac{dy}{dx} = \frac{du}{dx}.$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \frac{du}{dx}$$

$$= \frac{1}{\sqrt{1 - \sin^2 y}} \frac{du}{dx}.$$

The positive sign of the radical is chosen because $\cos y$ is positive when y lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, which has been taken to be the case in the definition of the principal value of $y = \sin^{-1} u$.

Hence

$$\frac{dy}{dx} = \frac{\frac{du}{dx}}{\sqrt{1 - u^2}}.$$

Therefore,

$$\frac{\mathbf{d(\sin^{-1} u)}}{\mathbf{du}} = \frac{\frac{\mathbf{du}}{\mathbf{dx}}}{\sqrt{\mathbf{1 - u^2}}}, \qquad \text{or} \quad \mathbf{d(\sin^{-1} u)} = \frac{\mathbf{du}}{\sqrt{\mathbf{1 - u^2}}}. \tag{1}$$

The student will show that

$$\frac{\mathbf{d(\cos^{-1} u)}}{\mathbf{dx}} = -\frac{\frac{\mathbf{du}}{\mathbf{dx}}}{\sqrt{\mathbf{1 - u^2}}}, \qquad \text{or} \quad \mathbf{d(\cos^{-1} u)} = -\frac{\mathbf{du}}{\sqrt{\mathbf{1 - u^2}}}, \tag{2}$$

$$\frac{\mathbf{d(\tan^{-1} u)}}{\mathbf{dx}} = \frac{\frac{\mathbf{du}}{\mathbf{dx}}}{\mathbf{1 + u^2}}, \qquad \text{or} \quad \mathbf{d(\tan^{-1} u)} = \frac{\mathbf{du}}{\mathbf{1 + u^2}}, \tag{3}$$

$$\frac{\mathbf{d(\cot^{-1} u)}}{\mathbf{dx}} = -\frac{\frac{\mathbf{du}}{\mathbf{dx}}}{\mathbf{1 + u^2}}, \qquad \text{or} \quad \mathbf{d(\cot^{-1} u)} = -\frac{\mathbf{du}}{\mathbf{1 + u^2}}, \tag{4}$$

$$\frac{\mathbf{d(\sec^{-1} u)}}{\mathbf{dx}} = \frac{\frac{\mathbf{du}}{\mathbf{dx}}}{\mathbf{u\sqrt{u^2 - 1}}}, \qquad \text{or} \quad \mathbf{d(\sec^{-1} u)} = \frac{\mathbf{du}}{\mathbf{u\sqrt{u^2 - 1}}}, \tag{5}$$

$$\frac{d(\csc^{-1} u)}{dx} = -\frac{\frac{du}{dx}}{u\sqrt{u^2 - 1}}, \quad \text{or} \quad d(\csc^{-1}u) = -\frac{du}{u\sqrt{u^2 - 1}}. \tag{6}$$

Illustration 1. If $y = \sin^{-1}(x^2 - 2x - 3)$, find dy. By formula (1)

$$dy = \frac{d(x^2 - 2x - 3)}{\sqrt{1 - (x^2 - 2x - 3)^2}}$$
$$= \frac{2(x - 1)dx}{\sqrt{1 - (x^2 - 2x - 3)^2}}.$$

Illustration 2. If $y = \tan^{-1} 3x$, find dy. By formula (3)

$$dy = \frac{d(3x)}{1 + (3x)^2}$$
$$= \frac{3dx}{1 + 9x^2}.$$

Exercises

Find $\frac{dy}{dx}$ in Exercises 1 to 10.

1. $y = \sin^{-1} \frac{x}{3}$.

2. $y = \sin^{-1} (x - 1)$.

3. $y = \tan^{-1} \frac{x}{2}$.

4. $y = \tan^{-1} (x + 1)$.

5. $y = \sec^{-1} 2x$.

6. $y = \tan^{-1} x^2$.

7. $y = x \tan^{-1} x$.

8. $y = x \sin^{-1} x$.

9. $y = \sin^{-1} (2x - 3)$.

10. $y = \tan^{-1} \frac{x + 1}{2}$.

11. The sine of an angle is given as 0.732 with a possible error of 0.0005. How accurately (approximately) is the angle known?

12. The tangent of an angle is given as 0.434 with a possible error of 0.0005. How accurately is the angle known?

83. Integrals Leading to Inverse Circular Functions.

Illustration 1. If $dy = \frac{dx}{1 + x^2}$, find y.

$$y = \int \frac{dx}{1 + x^2},$$

or

$$y = \tan^{-1} x + C.$$

Illustration 2. If $dy = \dfrac{dx}{1 + 9x^2}$, find y.

$$y = \int \frac{dx}{1 + 9x^2}$$

$$= \tfrac{1}{3} \int \frac{3dx}{1 + (3x)^2}.$$

The expression under the integral sign is now of the form $\dfrac{du}{1 + u^2}$ whose integral is $\tan^{-1} u$. Hence

$$y = \tfrac{1}{3} \tan^{-1} (3x) + C.$$

Illustration 3. If $dy = \dfrac{dx}{4 + 9x^2}$, find y.

$$y = \int \frac{dx}{4 + 9x^2}$$

$$= \tfrac{1}{4} \int \frac{dx}{1 + (\frac{3}{2}x)^2}$$

$$= \tfrac{1}{4} \cdot \tfrac{2}{3} \int \frac{\frac{3}{2}\, dx}{1 + (\frac{3}{2}x)^2}.$$

Hence

$$y = \tfrac{1}{6} \tan^{-1} (\tfrac{3}{2}x) + C.$$

Illustration 4. If $dy = \dfrac{dx}{\sqrt{4 - 9x^2}}$, find y.

$$y = \int \frac{dx}{\sqrt{4 - 9x^2}}$$

$$= \tfrac{1}{2} \int \frac{dx}{\sqrt{1 - (\frac{3}{2}x)^2}}$$

$$= \tfrac{1}{2} \cdot \tfrac{2}{3} \int \frac{\frac{3}{2}\, dx}{\sqrt{1 - (\frac{3}{2}x)^2}}.$$

The expression under the integral sign is now of the form $\dfrac{du}{\sqrt{1-u^2}}$ whose integral is $\sin^{-1} u$. Hence

$$y = \tfrac{1}{3}\sin^{-1}(\tfrac{3}{2}x) + C.$$

Integrals of the type which occur in *Illustrations* 3 and 4 are very common. The integration can be more simply performed by the use of the following formulas which are proved below.

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C. \tag{1}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C. \tag{2}$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C. \tag{3}$$

PROOF OF (1):

$$\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{\dfrac{du}{a}}{\sqrt{1-\dfrac{u^2}{a^2}}} = \int \frac{d\left(\dfrac{u}{a}\right)}{\sqrt{1-\left(\dfrac{u}{a}\right)^2}} = \sin^{-1}\frac{u}{a} + C.$$

PROOF OF (2):

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\int \frac{\dfrac{du}{a}}{1+\dfrac{u^2}{a^2}} = \frac{1}{a}\int \frac{d\left(\dfrac{u}{a}\right)}{1+\left(\dfrac{u}{a}\right)^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C.$$

PROOF OF (3):

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\int \frac{\dfrac{du}{a}}{\dfrac{u}{a}\sqrt{\dfrac{u^2}{a^2}-1}}$$

$$= \frac{1}{a}\int \frac{d\left(\dfrac{u}{a}\right)}{\dfrac{u}{a}\sqrt{\left(\dfrac{u}{a}\right)^2-1}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C.$$

Illustration 5.

$$\int \frac{dx}{\sqrt{9 - 4x^2}}.$$

Here $u = 2x$, $a = 3$.

$$\int \frac{dx}{\sqrt{9 - 4x^2}} = \tfrac{1}{2}\int \frac{2dx}{\sqrt{9 - 4x^2}} = \tfrac{1}{2}\sin^{-1}\frac{2x}{3} + C.$$

Illustration 6.

$$\int \frac{dx}{9 + 16x^2}.$$

Here $u = 4x$, $a = 3$.

$$\int \frac{dx}{9 + 16x^2} = \tfrac{1}{4}\int \frac{4dx}{9 + 16x^2} = \tfrac{1}{4}\,\tfrac{1}{3}\tan^{-1}\frac{4x}{3} + C$$
$$= \tfrac{1}{12}\tan^{-1}\frac{4x}{3} + C.$$

Illustration 7.

$$\int \frac{dx}{x\sqrt{25x^2 - 4}}.$$

Here $u = 5x$, $a = 2$.

$$\int \frac{dx}{x\sqrt{25x^2 - 4}} = \int \frac{5dx}{5x\sqrt{25x^2 - 4}} = \frac{1}{2}\sec^{-1}\frac{5x}{2} + C.$$

Exercises

1. $\displaystyle\int \frac{dx}{4 + x^2}.$

2. $\displaystyle\int \frac{dx}{25 + 16x^2}.$

3. $\displaystyle\int \frac{dx}{x^2 + 9}.$

4. $\displaystyle\int \frac{dx}{\sqrt{4 - 9x^2}}.$

5. $\displaystyle\int \frac{dx}{x\sqrt{4x^2 - 9}}.$

6. $\displaystyle\int \frac{dx}{5x^2 + 11}.$

7. $\displaystyle\int \frac{dx}{x^2 + 7}.$

8. $\displaystyle\int \frac{dx}{x\sqrt{9x^2 - 16}}.$

9. $\displaystyle\int \frac{dx}{\sqrt{13 - x^2}}.$

10. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 25}}.$

11. $\displaystyle\int \frac{x\,dx}{\sqrt{1 - x^4}}.$

12. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 19}}.$

13. $\displaystyle\int \frac{dx}{x\sqrt{3x^2 - 14}}.$

14. $\displaystyle\int \frac{dx}{\sqrt{25 - 9x^2}}.$

15. $\displaystyle\int \frac{dx}{4x^2 + 9}.$

16. $\displaystyle\int \frac{dx}{3x^2 + 7}.$

17. $\displaystyle\int \frac{dx}{\sqrt{4 - 25x^2}}.$

18. $\displaystyle\int \frac{dx}{\sqrt{5 - 3x^2}}.$

19. $\displaystyle\int \frac{dx}{\sqrt{16 - 9x^2}}.$

20. $\displaystyle\int \frac{dx}{\sqrt{9 - x^2}}.$

21. $\displaystyle\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{(x + 2)^2 + 1}.$

22. $\displaystyle\int \frac{dx}{x^2 + 6x + 25}.$

23. $\displaystyle\int \frac{dx}{x\sqrt{9x^2 - 1}}.$

24. $\displaystyle\int \frac{dx}{\sqrt{25 - 16x^2}}.$

25. $\displaystyle\int \frac{dx}{5x^2 + 8}.$

26. Find the area between the ordinates $x = 0$, $x = \frac{1}{2}$, the X-axis, and the curve $y = \dfrac{1}{\sqrt{1 - x^2}}$.

27. The slope of a curve at any point (x, y) is equal to $\dfrac{1}{1 + x^2}$. Find the equation of the curve if it passes through the point $(1, 0)$.

28. Find the area under the curve $y = \dfrac{8a^3}{x^2 + 4a^2}$, above the X-axis, and between the ordinates $x = 0$ and $x = b$. Find the limit of the area as b increases without limit.

29. Find the mean ordinate of the curve $y = \frac{1}{1 + x^2}$, between the limits $x = 0$ and $x = 1$.

30. Find the mean ordinate of the curve $y = \frac{1}{\sqrt{a^2 - x^2}}$, between $x = 0$ and $x = \frac{a}{2}$.

31. Find the mean distance from the Y-axis of the area under

$$y = \frac{1}{\sqrt{a^2 - x^2}},$$

between $x = 0$ and $x = \frac{a}{2}$.

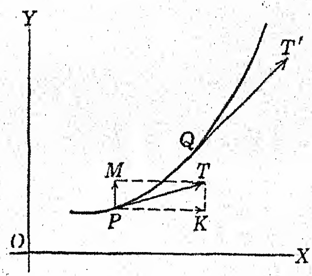

FIG. 58.

84. Velocity and Acceleration. The velocity of a particle moving in a curved path is, at any point, represented by a vector whose direction is that of the tangent to the curve at that point

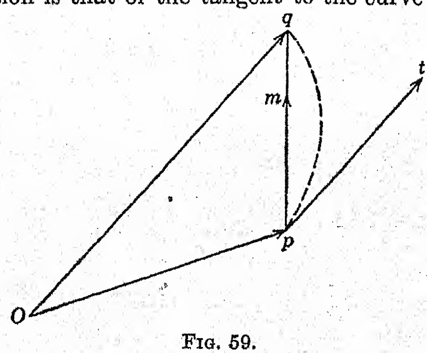

FIG. 59.

and whose length is equal to $\frac{ds}{dt}$. Thus the velocity at the point P, Fig. 58, is represented by the vector PT. Its components PK and PM parallel to the X- and Y-axes, respectively, are given by

$$\begin{aligned} PK &= v_x = \frac{dx}{dt}, \\ PM &= v_y = \frac{dy}{dt}. \end{aligned} \tag{1}$$

Further

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \tag{2}$$

In Fig. 58, let PT be the velocity at P, and QT' that at Q. Draw from a common origin, o, Fig. 59, the vectors op and oq equal to the vectors PT and QT', respectively. Then pq equals the vector increment, Δv. The average acceleration for the interval Δt is equal to $\frac{\Delta v}{\Delta t}$ directed along pq. Lay off, on pq, pm equal to $\frac{\Delta v}{\Delta t}$. The line pm is a vector representing the average acceleration for the interval Δt. As Δt approaches zero, Q approaches P, and q approaches p as indicated by the dotted line, Fig. 59; pm approaches a vector pt directed along the tangent to the arc pq at p. This vector, the limit of $\frac{\Delta v}{\Delta t}$, represents the acceleration of the particle moving in the curved path. Let us calculate its x and y components. In Fig. 59, denote:

op by v and its components by v_x and v_y,
oq by v' and its components by v_x' and v_y',
pq by Δv and its components by $\Delta v_x = v_x' - v_x$ and $\Delta v_y = v_y' - v_y$,
pt by j and its components by j_x and j_y.

Then

$$j_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \frac{dv_x}{dt} = \frac{d\left(\frac{dx}{dt}\right)}{dt} = \frac{d^2x}{dt^2}, \tag{3}$$

$$j_y = \lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t} = \frac{dv_y}{dt} = \frac{d\left(\frac{dy}{dt}\right)}{dt} = \frac{d^2y}{dt^2}. \tag{4}$$

The magnitude and direction of the vector j are given by:

$$j = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} \tag{5}$$

and

$$\tan \phi = \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}},$$

where ϕ is the angle made by pt, Fig. 59, with the positive direction of the X-axis.

85. Angular Velocity and Acceleration. If a body is rotating about an axis, the amount of rotation is given by the angle θ through which a line in the body turns which intersects the axis and is perpendicular to it. Thus in the case of a wheel the rotation is measured by the angle θ through which a spoke turns. θ is a function of the time t. The rotation is uniform if the body rotates through equal angles in equal intervals of time. If the uniform rate of rotation is ω radians per second, the body rotates through $\theta = \omega t$ radians in t seconds. If the rotation is not uniform, the rate at which the body is rotating at any instant, the angular velocity, is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}.$$

Similarly, the angular acceleration α is the time rate of change of the angular velocity. Then,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

If we consider a particle at a distance r from the axis of rotation, its linear velocity v is

$$v = \omega r$$

and is directed along the tangent to the circle described by the particle. The tangential acceleration is

$$j_t = \alpha r.$$

Illustration. A wheel of radius r, Fig. 60, is rotating with constant angular velocity ω. Find the directions and magnitudes of the velocity and the acceleration of a point on its rim.

Let P be a point on the rim and let OP make an angle $\theta = \omega t$ with OX. Then

$$x = r \cos \omega t$$
$$y = r \sin \omega t$$

and

$$\frac{dx}{dt} = -r\omega \sin \omega t = -\omega y$$
$$\frac{dy}{dt} = r\omega \cos \omega t = \omega x.$$

These components of the velocity are represented by PK and PM respectively. The magnitude of the resultant PN is given by

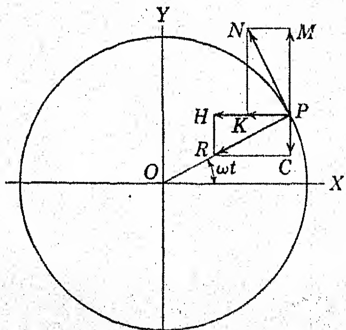

FIG. 60.

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \omega r,$$

while its direction follows from the relation

$$\frac{dy}{dx} = \frac{\omega r \cos \omega t}{-\omega r \sin \omega t} = -\cot \omega t.$$

It follows that PN is perpendicular to OP, as indicated in the figure. This direction also follows from

$$\frac{dy}{dx} = \frac{\omega x}{-\omega y} = -\frac{x}{y}.$$

The components of the acceleration are

$$j_x = \frac{d^2x}{dt^2} = -\omega^2 r \cos \omega t = -\omega^2 x$$
$$j_y = \frac{d^2y}{dt^2} = -\omega^2 r \sin \omega t = -\omega^2 y.$$

These components are represented by the lines PH and PC, respectively, in Fig. 60. The magnitude of the resultant accelera-

tion PR is equal to $\sqrt{j_x^2 + j_y^2} = \omega^2 r$. That the resultant is directed along the radius and toward the center follows from the relation

$$\frac{j_y}{j_x} = \frac{-\omega^2 r \sin \omega t}{-\omega^2 r \cos \omega t} = \tan \omega t = \frac{-\omega^2 y}{-\omega^2 x} = \frac{y}{x}.$$

Exercises

1. Establish the following formulas for:

(*a*) motion in a straight line with constant acceleration

$$\begin{aligned} v &= v_o + jt. \\ s &= v_o t + \tfrac{1}{2} j t^2. \\ \frac{v^2}{2} - \frac{v_o^2}{2} &= js. \end{aligned}$$

(*b*) rotation with constant angular acceleration

$$\begin{aligned} \omega &= \omega_o + \alpha t. \\ \theta &= \omega_o t + \tfrac{1}{2}\alpha t^2. \\ \frac{\omega^2}{2} - \frac{\omega_o^2}{2} &= \alpha\theta. \end{aligned}$$

2. A flywheel 10 feet in diameter makes 25 revolutions a minute. What is the linear velocity of a point on the rim?

3. Find the constant acceleration, such as the retardation caused by a brake, which would bring the wheel of Exercise 2 to rest in 30 seconds. How many revolutions would it make before coming to rest?

4. A resistance retards the motion of a wheel at the rate of 0.5 radian per second per second. If the wheel is running at the rate of 10 revolutions a second when the resistance begins to act, how many revolutions will it make before stopping?

5. A wheel of radius r is rolling with the uniform angular velocity ω along a horizontal surface without slipping. How fast is the axle moving forward? The parametric equations of a point P on the rim are:

$$\begin{aligned} x &= r(\omega t - \sin \omega t) \\ y &= r(1 - \cos \omega t). \end{aligned}$$

Find the magnitude and the direction of the velocity of P at any instant. What is the velocity of a point at the top of the wheel? At the bottom?

6. If a particle moves in such a way that its coordinates are $x = a\cos t + b$, $y = a\sin t + c$, where t denotes time, find the equation of the path and show that the particle moves with constant speed.

7. A particle moves on the curve $y = x^2$ with a constant speed of 2 units per second, and in a direction such that x increases with the time. Find in magnitude and direction the acceleration of the particle when $x = 1$.

86. Simple Harmonic Motion. Let the point P, Fig. 61, move upon the circumference of a circle of radius a feet with the uniform velocity of v feet per second, so that the radius OP rotates at the rate of $\frac{v}{a} = \omega$ radians per second. The projection, B, of P on the vertical diameter moves up and down. If the point P was at C when $t = 0$, the displacement, $OB = y$, is given by

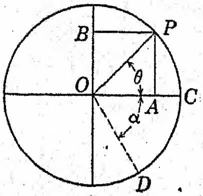

FIG. 61.

$$y = a\sin\theta = a\sin\omega t.$$

If the point P was at D when $t = 0$, we have

$$y = a\sin(\omega t - \alpha). \tag{1}$$

Any motion such that the displacement at time t is given by (1) is called a simple harmonic motion. Thus the point B, Fig. 61, describes simple harmonic motion.

From (1) it follows that the velocity of a point describing simple harmonic motion is

$$\frac{dy}{dt} = a\omega\cos(\omega t - \alpha) \tag{2}$$

and that the acceleration is

$$\frac{d^2y}{dt^2} = -a\omega^2\sin(\omega t - \alpha). \tag{3}$$

The second member is $-\omega^2 y$, by equation (1). Hence

$$\frac{d^2y}{dt^2} = -\omega^2 y, \tag{4}$$

or

$$\frac{d^2y}{dt^2} + \omega^2 y = 0. \tag{5}$$

Equation (4) shows that the acceleration of a particle describing simple harmonic motion is proportional to the displacement and oppositely directed. That the acceleration is oppositely directed to the displacement is to be expected from the character of the motion, which is an oscillation about a position of equilibrium. Thus if the body is above this position, the force is directed downward and *vice versa*. In Fig. 61, the point B has a positive acceleration when below O and a negative acceleration when above O. The acceleration is zero at O and reaches its greatest numerical values at the upper and lower ends of the diameter.

The velocity has its greatest numerical value when B passes through O in either direction.

Equation (4), or (5), is called the differential equation of simple harmonic motion. The proportionality factor ω^2 is connected with the period T by the relation

$$T = \frac{2\pi}{\omega}.$$

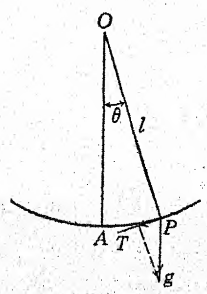

Fig. 62.

Frequently it is desired to solve the converse problem, *viz.*, to find the motion of a particle whose acceleration is proportional to the displacement and oppositely directed. In other words, a relation between y and t is sought which satisfies equation (4). Clearly (1) is such a relation. However, it will be instructive to obtain this relation directly from (4).

First, a differential equation equivalent to (4) will be obtained in the solution of the problem of the motion of the simple pendulum.

87. The Simple Pendulum. Let P, Fig. 62, be a position of the bob of a simple pendulum at a given instant and let it be moving to the right. If s denotes the displacement considered positive on the right of the position of equilibrium, $\frac{d^2s}{dt^2}$ is the acceleration in the direction of the tangent PT, for $\frac{ds}{dt}$ is the velocity along the tangent. This acceleration must be equal to the tangential com-

ponent of the acceleration due to gravity, if the resistance of the air be neglected. This component is equal to $-g \sin \theta$. Since it acts in a direction opposite to that in which s is increasing, it must be taken with the negative sign. We have then

$$\frac{d^2s}{dt^2} = -g \sin \theta. \tag{1}$$

If the angle through which the pendulum swings is small, $\sin \theta$ can be replaced by θ. Then (1) becomes

$$\frac{d^2s}{dt^2} = -g\theta. \tag{2}$$

Since $s = l\theta$,

$$\frac{d^2\theta}{dt^2} = -\frac{g\theta}{l}. \tag{3}$$

Putting $\frac{g}{l} = \omega^2$ for convenience in writing,

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta, \tag{4}$$

an equation of the form of (4), §86.

Multiplying by $2\frac{d\theta}{dt}$ and integrating,

$$\left(\frac{d\theta}{dt}\right)^2 = -\omega^2\theta^2 + C^2.$$

The arbitrary constant is written for convenience in the form C^2. The constant must be positive. Otherwise the velocity would be imaginary. Extracting the square root, and retaining only the positive sign,

$$\frac{d\theta}{dt} = \sqrt{C^2 - \omega^2\theta^2},$$

or

$$\frac{d\theta}{\sqrt{C^2 - \omega^2\theta^2}} = dt. \tag{5}$$

Integration gives

$$\frac{1}{\omega}\sin^{-1}\frac{\omega\theta}{C} = t + C_1.$$

$$\sin^{-1}\frac{\omega\theta}{C} = \omega t + \omega C_1$$
$$= \omega t + C_2,$$

where the constant ωC_1 is replaced by the constant C_2. Then

$$\frac{\omega\theta}{C} = \sin(\omega t + C_2),$$

$$\theta = \frac{C}{\omega}\sin(\omega t + C_2)$$
$$= C_3 \sin(\omega t + C_2),$$

where $\frac{C}{\omega}$ has been replaced by C_3. Therefore

$$\theta = C_3 \sin(\omega t + C_2) \tag{6}$$

is the equation of the angular displacement of the pendulum.

The form of (6) shows that the motion is of period $\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$. It is a simple harmonic motion and contains two arbitrary constants. They can be determined by two conditions, *e.g.*, the displacement and velocity at a given instant. Suppose the bob drawn aside to the right so that the string makes an angle θ_0 with the vertical. The bob is then released without being given an impulse; *i.e.*, with an initial velocity zero. The time will be counted from the instant of release. The conditions are then

$$\theta = \theta_0 \tag{7}$$

and

$$\frac{d\theta}{dt} = 0 \tag{8}$$

when $t = 0$. From (6),

$$\frac{d\theta}{dt} = \omega C_3 \cos(\omega t + C_2).$$

The condition (8) gives

$$0 = \omega C_3 \cos C_2,$$

or $\cos C_2 = 0$. Whence $C_2 = \dfrac{\pi}{2}$. Then (6) becomes

$$\theta = C_3 \cos \omega t.$$

The condition (7) gives

$$\theta_0 = C_3.$$

Hence

$$\theta = \theta_0 \cos \omega t. \tag{9}$$

Multiplying by l and recalling that $l\theta = s$, and denoting $l\theta_0$ by s_0, we have as the equation for the displacement s,

$$s = s_0 \cos \omega t. \tag{10}$$

The same result would have been obtained if the minus sign had been used before the radical in (5) and in the equation preceding it. The period is $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{l}{g}}$. When is the velocity of the bob greatest? When least, numerically?

Equation (6), the solution of (4), shows that, if the acceleration of a particle is proportional to its displacement and oppositely directed, the particle describes simple harmonic motion.

Exercises

1. Write the differential equations of the following simple harmonic motions. Find the period in each case.

$$y = 5 \sin 3t.$$
$$y = 6 \sin \left(3t + \frac{\pi}{3}\right).$$
$$y = 5 \cos 3t.$$
$$y = 4 \sin 2t + 3 \cos 2t.$$
$$y = 7 \sin (8t + \alpha).$$

2. Write the equation of a simple harmonic motion which satisfies the equations:

$$\frac{d^2y}{dt^2} + 9y = 0.$$

$$\frac{d^2y}{dt^2} + 3y = 0.$$

$$\frac{d^2y}{dt^2} + a^2y = 0.$$

CHAPTER IX

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

88. Derivative of the Exponential and Logarithmic Functions. Let

$$y = a^x. \tag{1}$$

Then

$$\begin{aligned} y + \Delta y &= a^{x+\Delta x} \\ \Delta y &= a^x(a^{\Delta x} - 1) \\ \frac{\Delta y}{\Delta x} &= a^x\left(\frac{a^{\Delta x} - 1}{\Delta x}\right) \\ \frac{dy}{dx} &= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}. \end{aligned} \tag{2}$$

Since $\dfrac{a^{\Delta x} - 1}{\Delta x}$ is independent of x, $\lim\limits_{\Delta x \to 0} \dfrac{a^{\Delta x} - 1}{\Delta x}$ is a constant for a given value of a. Call this constant K, so that

$$K = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}. \tag{3}$$

Then, from (2),

$$\frac{dy}{dx} = Ka^x. \tag{4}$$

Equation (4) shows that the slope of the curve $y = a^x$ is proportional to the ordinate of the curve. In other words, the rate of increase of the exponential function is proportional to the function itself.

When $x = 0$, it follows from (2) and (3) that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = K.$$

Consequently, the constant K introduced above is the slope of the curve $y = a^x$ at the point (0, 1). This slope depends upon the value of a. It will range from zero when $a = 1$, to very large values when a is large. Consideration of the curves $y = 2^x$ and $y = 3^x$ will show that the slope of the first at (0, 1) is less than 1 while that of the second is greater than 1. Let e be that value of a for which the corresponding curve, $y = e^x$, has a slope equal to 1 at the point where it crosses the Y-axis.

If, then,

$$y = e^x, \tag{5}$$

equation (4) becomes

$$\frac{dy}{dx} = e^x,$$

since $K = 1$ in this case. Or

$$\frac{de^x}{dx} = e^x. \tag{6}$$

Then

$$\mathbf{\frac{de^u}{dx} = e^u\frac{du}{dx}} \tag{7}$$

and

$$\mathbf{de^u = e^u\, du.} \tag{8}$$

Equation (6) shows that the slope of the curve $y = e^x$ is equal to the ordinate of the curve. The number e is the base of the natural system of logarithms. It is sometimes called the Napierian base. Its value, 2.71828 . . . , will be calculated later in §169.

The formula for the derivative of the natural logarithm of a function is now easily obtained. *In calculus if no base is indicated, the natural base, e, is understood.* Thus log u means $\log_e u$.

If

$$y = \log u,$$
$$u = e^y$$

and by (7)

$$\frac{du}{dx} = e^y \frac{dy}{dx}.$$

Whence

$$\frac{dy}{dx} = \frac{1}{e^y}\frac{du}{dx}$$
$$= \frac{1}{u}\frac{du}{dx}.$$

That is,

$$\frac{\mathbf{d}(\log \mathbf{u})}{\mathbf{dx}} = \frac{\mathbf{1}}{\mathbf{u}}\frac{\mathbf{du}}{\mathbf{dx}},$$

or

$$\mathbf{d}(\log \mathbf{u}) = \frac{\mathbf{du}}{\mathbf{u}}. \tag{9}$$

Since[1]

$$\log_a u = \log_a e \log u, \tag{10}$$

it follows from (9) that

$$\frac{\mathbf{d}\,(\log_a \mathbf{u})}{\mathbf{dx}} = \log_a \mathbf{e}\,\frac{\mathbf{1}}{\mathbf{u}}\frac{\mathbf{du}}{\mathbf{dx}},$$

or (11)

$$\mathbf{d}(\log_a \mathbf{u}) = \log_a \mathbf{e}\,\frac{\mathbf{du}}{\mathbf{u}}.$$

If $y = a^u$,

$$\log y = u \log a$$
$$\frac{1}{y}\frac{dy}{dx} = \log a \frac{du}{dx}$$

[1] Let $z = \log u.$
Then $e^z = u.$
Taking logarithms to the base a,
$$z \log_a e = \log_a u.$$
That is,
$$\log u \log_a e = \log_a u.$$

$$\frac{dy}{dx} = y \log a \frac{du}{dx}$$

$$= a^u \log a \frac{du}{dx}.$$

That is,

$$\mathbf{\frac{da^u}{dx} = a^u \log a \frac{du}{dx}}$$

or (12)

$$\mathbf{da^u = a^u \log a\, du.}$$

Illustrations.

1. If $y = e^{x^2}$, $dy = e^{x^2}d(x^2) = 2xe^{x^2}\,dx.$
2. If $y = e^{\sin x}$, $dy = e^{\sin x}\,d(\sin x) = \cos x\, e^{\sin x}\,dx.$
3. If $y = \log_{10}(x + 1)$, $dy = \log_{10} e \dfrac{d(x+1)}{x+1} = \log_{10} e \dfrac{dx}{x+1}.$
4. If $y = \log(x^2 + 1)$, $dy = \dfrac{2x\,dx}{x^2+1}.$
5. If $y = \log\sqrt{x^2 + 9} = \frac{1}{2}\log(x^2 + 9)$, $dy = \frac{1}{2}\dfrac{2x\,dx}{x^2+9}$

$$= \frac{x\,dx}{x^2+9}.$$

6. If $y = \log\dfrac{(1+x)^2}{(1-x)^3}$, $y = 2\log(1 + x) - 3\log(1 - x)$,

and

$$dy = \frac{2dx}{1+x} + \frac{3dx}{1-x} = \frac{5+x}{1-x^2}dx.$$

7. If $y = e^x \sin x$, $\dfrac{dy}{dx} = e^x(\cos x + \sin x)$

and

$$\frac{d^2y}{dx^2} = 2e^x \cos x.$$

5.
$$\int \tan u\, du = -\int -\frac{\sin u}{\cos u}\, du$$
$$= -\log \cos u + C$$
$$= \log \sec u + C.$$

Hence

$$\int \mathbf{\tan u\, du = \log \sec u + C.} \qquad (1)$$

6.
$$\int \cot u\, du = \int \frac{\cos u}{\sin u}\, du.$$

Hence

$$\int \mathbf{\cot u\, du = \log \sin u + C.} \qquad (2)$$

7.
$$\int \sec u\, du = \int \frac{(\sec u + \tan u) \sec u\, du}{\sec u + \tan u}$$
$$= \int \frac{\sec u \tan u + \sec^2 u}{\sec u + \tan u}\, du.$$

Hence

$$\int \mathbf{\sec u\, du = \log (\sec u + \tan u) + C.} \qquad (3)$$

8.
$$\int \csc u\, du = \int \frac{\csc u \cot u + \csc^2 u}{\csc u + \cot u}\, du.$$

Hence

$$\int \mathbf{\csc u\, du = -\log (\csc u + \cot u) + C.} \qquad (4)$$

9.
$$\int \frac{du}{\sqrt{u^2 + a^2}}$$

Let $u = a \tan \theta$.

$$du = a \sec^2 \theta\, d\theta$$
$$\sqrt{u^2 + a^2} = a\sqrt{\tan^2 \theta + 1} = a \sec \theta$$
$$\int \frac{du}{\sqrt{u^2 + a^2}} = \int \frac{a \sec^2 \theta\, d\theta}{a \sec \theta} = \int \sec \theta\, d\theta$$
$$= \log (\sec \theta + \tan \theta) + C_1.$$

From $\tan\theta = \frac{u}{a}$, we obtain $\sec\theta = \sqrt{\frac{u^2+a^2}{a^2}}$.

Then

$$\int \frac{du}{\sqrt{u^2+a^2}} = \log\left(\frac{\sqrt{u^2+a^2}}{a} + \frac{u}{a}\right) + C_1$$
$$= \log\left(u + \sqrt{u^2+a^2}\right) + C_1 - \log a.$$

Hence

$$\int \frac{\mathbf{du}}{\sqrt{\mathbf{u^2+a^2}}} = \mathbf{\log\left(u + \sqrt{u^2+a^2}\right) + C.} \tag{5}$$

10. $$\int \frac{du}{\sqrt{u^2-a^2}}.$$

Let

$$u = a\sec\theta.$$
$$du = a\sec\theta\tan\theta\, d\theta.$$

Then

$$\int \frac{du}{\sqrt{u^2-a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{a\tan\theta} = \int \sec\theta\, d\theta$$
$$= \log(\sec\theta + \tan\theta) + C_1.$$

From $\sec\theta = \frac{u}{a}$, we obtain $\tan\theta = \sqrt{\frac{u^2-a^2}{a^2}}$.

Then

$$\int \frac{du}{\sqrt{u^2-a^2}} = \log\left(\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right) + C_1$$
$$= \log\left(u + \sqrt{u^2-a^2}\right) + C_1 - \log a.$$

Hence

$$\int \frac{\mathbf{du}}{\sqrt{\mathbf{u^2-a^2}}} = \mathbf{\log\left(u + \sqrt{u^2-a^2}\right) + C.} \tag{6}$$

The results (5) and (6) may be combined in the single formula

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log\left(u + \sqrt{u^2 \pm a^2}\right) + C. \tag{7}$$

Exercises

In a few of the following exercises the integration can be performed by the methods of earlier sections, but not by the methods of this section.

1. $\int x^2 e^{x^3}\, dx.$
2. $\int e^{\tan x} \sec^2 x\, dx.$
3. $\int \frac{dx}{x+1}.$
4. $\int \frac{dx}{1-x}.$
5. $\int \frac{x\, dx}{1+x^2}.$
6. $\int e^{3x}\, dx.$
7. $\int \frac{e^x + 1}{e^x + x}\, dx.$
8. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx.$
9. $\int \frac{\cos x\, dx}{\sin x + 3}.$
10. $\int (\log x)^3 \frac{dx}{x}.$
11. $\int \frac{2-x}{3+4x-x^2} dx.$
12. $\int \frac{2x-3}{x^2-3x+5}\, dx.$
13. $\int \frac{(x+3)dx}{x^2+6x+7}.$
14. $\int e^{\sin 2x} \cos 2x\, dx.$
15. $\int 5^x\, dx.$
16. $\int \frac{x^2\, dx}{a^3 + x^3}.$
17. $\int \frac{x^2\, dx}{\sqrt{a^3 + x^3}}.$
18. $\int \frac{x\, dx}{\sqrt{1+x^2}}.$
19. $\int 4^{3x}\, dx.$
20. $\int \frac{dx}{\sqrt{9-x^2}}.$
21. $\int \sin\theta\, e^{(1+\cos\theta)} d\theta.$
22. $\int \frac{(3x+2)dx}{3x^2+4x+9}.$
23. $\int \frac{e^{2x}-1}{e^{2x}+1} dx.$
24. $\int \frac{dx}{\cos^2(3x-2)}.$

25. $\int_1^x \frac{dx}{x}$.

26. $\int_0^{\frac{\pi}{2}} \frac{1 - \sin x}{x + \cos x}\,dx$.

27. $\int \tan 3\theta \, d\theta$.

28. $\int \tan 4x \, dx$.

29. $\int \csc 5x \, dx$.

30. $\int \csc 3x \, dx$.

31. $\int \frac{dx}{\sqrt{x^2 + 9}}$.

32. $\int \frac{dx}{\sqrt{x^2 - 4}}$.

33. $\int \frac{dx}{\sqrt{4x^2 + 5}}$.

34. $\int \sec 2x \, dx$.

35. $\int \frac{dx}{\sqrt{9x^2 + 4}}$.

36. $\int \sec (2x - 3)dx$.

37. $\int \csc (3x - 4)dx$.

38. $\int \frac{dx}{\sqrt{2x^2 - 3}}$.

39. $\int \frac{dx}{\sqrt{x^2 + 16}}$.

40. $\int \csc (2x - 5)dx$.

41. $\int \cot (7 - 5x)dx$.

42. $\int \frac{(x - 3)}{\sqrt{x^2 - 6x + 7}}\,dx$.

43. $\int \frac{dx}{\sqrt{9x^2 - 25}}$.

44. Find the area bounded by $xy = 2$, $x = 2$, $x = 4$, and the X-axis.

45. Gas within a cylinder is being compressed by a movable piston in accordance with Boyle's law, $pv = C$. Find the work done in reducing the volume from 4 cubic feet at a pressure of 5000 pounds per square foot to a volume of 2 cubic feet. Calculate the result correct to five significant figures.

46. The gas within a cylinder 20 inches in diameter is being compressed by a movable piston in accordance with Boyle's law, $pv = C$. Find the work done when the piston moves a distance of 10 inches, if the initial pressure and volume were, respectively, 100 pounds per square inch and 12 cubic feet.

47. The slope of a curve at any point is equal to one-half of the ordinate at that point. Find the equation of the curve if it passes through the point (2, 1).

48. The slope of a curve at any point is proportional to the ordinate of that point. Find the equation of the curve if it passes through the point (1, 3) and has a slope of 2 at this point. Find the equation of the curve if it passes through the point (1, 2) and has a slope equal to 2 when $x = 3$.

49. Find the mean ordinate of the curve $y = \frac{1}{x}$ between $x = 1$ and $x = 3$.

50. Find the length of $y = \frac{1}{2}(e^x + e^{-x})$ between $x = 0$ and $x = 1$.

51. Find the area completely bounded by $y = e^x + e^{-x}$, $y = e^x - e^{-x} + 2$, and $x = 1$.

52. Find the area of the surface generated by revolving about the X-axis the part of the curve $y = \frac{1}{2}(e^x + e^{-x})$ which lies between $x = 0$ and $x = 1$.

53. Find the volume of the solid generated by the revolution of the curve of Exercise 52.

54. The subnormal of a curve is constant and equal to 3. Find the equation of the curve if it passes through the point (2, 5).

55. The subtangent of a curve is constant and equal to 3. Find the equation of the curve if it passes through the point (2, 5).

56. Find the mean ordinate of the curve $y = e^x$ between $x = 1$ and $x = 2$.

90. Logarithmic Differentiation. It is often advantageous in finding the derivative of $y = f(x)$ to take the logarithm of each member before differentiating A number of examples will be solved to illustrate the process.

Illustration 1. Find the derivative of $\frac{(x-1)^{\frac{2}{3}}}{(x+1)^{\frac{3}{5}}}$. Let

$$y = \frac{(x-1)^{\frac{2}{3}}}{(x+1)^{\frac{3}{5}}}$$

and take the logarithm of each member.

$$\log y = \tfrac{2}{3} \log (x-1) - \tfrac{3}{5} \log (x+1).$$

Differentiating,

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{3(x-1)} - \frac{3}{5(x+1)}$$

$$= \frac{x + 19}{15(x^2 - 1)}.$$

$$\frac{dy}{dx} = \frac{x + 19}{15(x^2 - 1)}y$$

$$= \frac{(x + 19)}{15(x^2 - 1)} \frac{(x - 1)^{\frac{2}{3}}}{(x + 1)^{\frac{3}{5}}}$$

$$= \frac{x + 19}{15(x - 1)^{\frac{1}{3}}(x + 1)^{\frac{8}{5}}}.$$

Illustration 2. Find the derivative of $\dfrac{\sqrt{1 - x^2}}{\sqrt[3]{x^2 + 1}}$. Let

$$y = \frac{\sqrt{1 - x^2}}{\sqrt[3]{x^2 + 1}}.$$

$$\log y = \tfrac{1}{2} \log (1 - x^2) - \tfrac{1}{3} \log (x^2 + 1)$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{x}{1 - x^2} - \frac{2x}{3(x^2 + 1)}$$

$$= -\frac{x(5 + x^2)}{3(1 - x^4)}$$

$$\frac{dy}{dx} = -\frac{x(5 + x^2)}{3(1 - x^4)} \frac{\sqrt{1 - x^2}}{\sqrt[3]{x^2 + 1}}$$

$$= -\frac{x(5 + x^2)}{3\sqrt{1 - x^2}(x^2 + 1)^{\frac{4}{3}}}.$$

This method is manifestly shorter and simpler than that of differentiating by the rule for the derivative of a quotient.

Illustration 3. Find the derivative of $(x^2 + 1)^{3x+2}$. Let

$$y = (x^2 + 1)^{3x+2}.$$

$$\log y = (3x + 2) \log (x^2 + 1)$$

$$\frac{1}{y}\frac{dy}{dx} = (3x + 2)\frac{2x}{x^2 + 1} + 3 \log (x^2 + 1)$$

$$\frac{dy}{dx} = \left[(3x + 2)\frac{2x}{x^2 + 1} + 3 \log (x^2 + 1)\right](x^2 + 1)^{3x+2}.$$

$\dfrac{1}{y}\dfrac{dy}{dx}$ is called the logarithmic derivative of y with respect to x. It will be considered further in a later article.

Exercises

Find the derivative in Exercises 1 to 8.

1. $y = \dfrac{(x+1)^{\frac{3}{7}}}{(x-7)^{\frac{2}{5}}}$.
2. $y = \dfrac{(x+3)^2}{(x-4)^3(x-5)^4}$.
3. $y = (x+1)^{\frac{2}{3}}(2x+5)^{\frac{3}{4}}$.
4. $y = x(1+x)\sqrt{1-x}$.
5. $y = x^n n^x$. (Solve by two methods.)
6. $y = x^{\sin x}$.
7. $s = (7t+3)10^{3t-2}$.
8. $y = x^{\sqrt{x}}$.

In Exercises 9 to 16 find the logarithmic derivative.

9. $y = e^{7x}$.
10. $y = x^2$.
11. $y = \dfrac{1}{x^2}$.
12. $y = x^n$.
13. $y = cx^n$.
14. $y = e^{kx+L} = ce^{kx}$.
15. $y = 10^{kx+L}$.
16. $y = uv$, where u and v are functions of x.
17. $y = uvw$, where u, v, and w are functions of x. Find $\dfrac{dy}{dx}$.

Find y if its logarithmic derivative is:

18. $6x + 7$. *Ans.* $y = Ce^{3x^2+7x}$
19. $\dfrac{3}{x}$.
20. $\dfrac{n}{x}$.
21. $\dfrac{1}{x+1}$.
22. $\dfrac{x^2}{1-x^3}$.
23. $\dfrac{3}{xy}$.

91. The Derivative of ax^n. The formula for the derivative of ax^n can be proved for all values of the constant n, by using logarithmic differentiation (see §29 where n was taken to be a positive integer in the proof).

Let

$$y = ax^n$$
$$\log y = \log a + n \log x$$
$$\frac{1}{y}\frac{dy}{dx} = n\frac{1}{x}. \qquad (1)$$

Then

$$\frac{dy}{dx} = n\frac{y}{x} = nax^{n-1}.$$

This proves the formula for any constant n.

Equation (1) can be written in the form

$$\frac{dy}{y} = n\frac{dx}{x}. \tag{2}$$

92. Compound Interest Law. If

$$y = Ce^{kt}, \tag{1}$$

$$\frac{dy}{dt} = Cke^{kt} = ky. \tag{2}$$

Equation (2) expresses the fact already noted in §88, as a characteristic property of the exponential function, *viz.*, that the function increases at a rate proportional to itself. We can show, conversely, that if a function increases at a rate proportional to itself, it is an exponential function.

Thus, let it be given that

$$\frac{dy}{dt} = ky. \tag{3}$$

Then

$$\frac{dy}{y} = k\,dt$$

$$\log y = kt + C$$

$$y = e^{kt+C} = e^{C}e^{kt}.$$

Hence

$$y = C_1e^{kt}. \tag{4}$$

When a function varies according to this law it is said to follow the "compound interest law," for the function increases in a way somewhat analogous to that in which a sum of money placed at compound interest increases. The rate of increase for an interest

term is proportional to the amount accumulated at the beginning of that term. As the accumulation grows, the rate of increase for an interest term grows. The law is also called the law of growth. For example, the population of a country, the number of bacteria in a culture, the size of a tree, all, for a time at least, follow this law very closely. In many cases in nature the function decreases at a rate proportional to itself. The compound interest law appears in this case in the form Ce^{-kt}, where k is a positive constant. For if

$$\frac{dy}{dt} = -ky,$$

it follows that

$$y = Ce^{-kt}.$$

The law in this case may be called the law of decay.

Illustration 1. Newton's law of cooling states that the temperature of a heated body surrounded by a medium of constant temperature decreases at a rate proportional to the difference in temperature between the body and the medium. Let θ denote the difference in temperature. Then

$$\frac{d\theta}{dt} = -k\theta \tag{5}$$

and

$$\theta = Ce^{-kt}. \tag{6}$$

The meaning of the constant C is seen at once on setting $t = 0$. It is the difference in temperature between the body and the medium at the time $t = 0$. If this initial difference in temperature is known, (6) gives the temperature of the body at any later instant. Call the initial temperature θ_0, the temperature of the medium being taken as the zero of the temperature scale. Equation (6) becomes

$$\theta = \theta_0 e^{-kt}. \tag{7}$$

The time which is required for the difference in temperature to fall from θ_1 to θ_2 can be found from (7). Thus

$$\begin{aligned}\theta_1 &= \theta_0 e^{-kt_1}\\ \theta_2 &= \theta_0 e^{-kt_2}\\ \frac{\theta_1}{\theta_2} &= e^{-k(t_1-t_2)},\end{aligned}$$

whence

$$t_2 - t_1 = \frac{1}{k}\log\frac{\theta_1}{\theta_2}. \tag{8}$$

This result could have been obtained directly from the differential equation (5). Thus

$$\begin{aligned}\frac{d\theta}{dt} &= -k\theta,\\ dt &= -\frac{1}{k}\frac{d\theta}{\theta}.\end{aligned} \tag{9}$$

Integrating,

$$t = -\frac{1}{k}\log\theta + C.$$

When

$$t = t_1, \quad \theta = \theta_1.$$

Then

$$C = t_1 + \frac{1}{k}\log\theta_1.$$

Hence

$$t - t_1 = \frac{1}{k}(\log\theta_1 - \log\theta).$$

In particular,

$$t_2 - t_1 = \frac{1}{k}\log\frac{\theta_1}{\theta_2}.$$

Illustration 2. Find the law of variation of the atmospheric pressure with height.

Consider a column of air of unit cross section, Fig. 63. Denote height above sea level by h and the pressure on unit cross section at this height by p. The difference in pressure at C and D is the weight of the gas within the element of volume of height Δh.

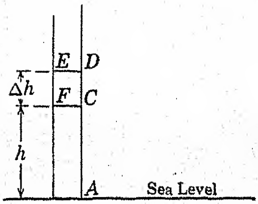

FIG. 63.

Thus

$$\Delta p = -g\rho\Delta h,$$

where ρ is the average density of the air in the volume $CDEF$. Then

$$\frac{\Delta p}{\Delta h} = -g\rho \tag{1}$$

and

$$\frac{dp}{dh} = -g\rho,$$

where ρ is the density at C. If the temperature is assumed constant, the air obeys Boyle's law, $pv = C$, where v denotes the volume occupied by unit mass of air. Since

$$\rho = \frac{\text{mass}}{\text{volume}} = \frac{1}{v} = \frac{p}{C}.$$

$$\frac{dp}{dh} = -kp,$$

where

$$k = \frac{g}{C}.$$

Integration gives

$$\log p = -kh + \log C_1,$$

or

$$p = C_1e^{-kh}.$$

When $h = 0$, $p = p_0$, the pressure at sea level, and $C_1 = p_0$. Hence

$$p = p_0e^{-kh}. \tag{2}$$

Exercises

1. Assuming that the retardation of a boat moving in still water is proportional to the velocity, find the velocity at the end of time t and the distance passed over in time t after the engine was shut off, if the boat was moving at the rate of 7 miles per hour at that time. *Ans.* $s = \frac{7}{k}(1 - e^{-kt})$.

2. The number of bacteria per cubic centimeter of culture increases under proper conditions at a rate proportional to the number present. Find an expression for the number present at the end of time t. Find the time required for the number per cubic centimeter to increase from b_1 to b_2. Does this time depend on the number present at the time $t = 0$?

3. A disk is rotating about a vertical axis in a liquid. If the retardation due to friction of the liquid is proportional to the angular velocity ω, find ω after t seconds if the initial angular velocity was ω_0.

4. If the disk of Exercise 3 is rotating very rapidly, the retardation is proportional to ω^2. Find ω after t seconds if the initial angular velocity was ω_0.

5. A law for the velocity of chemical reactions states that the amount of chemical change per unit of time is proportional to the mass of changing substance present in the system. The rate at which the change takes place is proportional to the mass of the substance still unchanged. If q denotes the original mass, find an expression for the mass remaining unchanged after a time t has elapsed.

6. A body is cooling according to Newton's law. θ is the difference between the temperature of the body and that of the medium. If θ falls from 60° to 50°C. in 150 seconds, find an expression for θ as a function of t, determining the numerical values of the constants. Find θ at the end of 300 seconds. How long will it take for θ to fall to 30°C.?

7. If at a certain instant the barometer reads 30 inches at sea level and 24 inches at 6000 feet above sea level, find the barometric reading

at 10,000 feet above sea level under the assumptions of *Illustration* 2 above.

93. Relative Rate of Increase. If the rate of change of a function is divided by the function itself, the quotient is the rate of change of the function per unit value of the function. This quotient has been called the *relative rate of increase* of the function. If a function varies according to the compound interest law, its relative rate of increase is constant, *i.e.*,

$$\frac{1}{y}\frac{dy}{dt} = k.$$

One hundred times the relative rate of increase is the per cent rate of increase. Thus if

$$\frac{1}{y}\frac{dy}{dt} = 0.02,$$

the per cent rate of increase is 2. This means that y increases 2 per cent per unit time. Any of the Exercises 1 to 7 of **§92** might have been stated in terms of the relative rate of increase of the function concerned.

Exercises

1. Given that the intensity of light is diminished 2 per cent by passing through 1 millimeter of glass, find the intensity I as a function of t, the thickness of the glass through which the light passes.

2. The temperature of a body cooling according to Newton's law fell from 30° to 18° in 6 minutes. Find the per cent rate of decrease of temperature per minute.

3. In the case of the rotating disk of Exercise 3, **§92**, the angular velocity is diminished at the constant (relative) rate of 3 per cent per second. If the disk revolved initially at the rate of two revolutions per second, find the rate at which it is revolving at the end of 10 seconds. How long will it take for the rate to become 1 revolution per second?

4. The linear coefficient k of thermal expansion of a bar is the increase in the length l per unit length per degree increase in temperature θ. Show that

$$\frac{1}{l}\frac{dl}{d\theta} = k.$$

Find the percentage increase in length of a bar if its temperature is increased 100°C., if $k = 0.00001$.

5. The radius of a sphere is increasing at a certain rate. Compare the relative rates of increase of the volume and of the radius.

6. The sides of a square are increasing at a certain rate. Compare the relative rates of increase of the area and of the sides.

7. The amplitude of a pendulum swinging in a resisting medium decreases from 5 to 3 inches in 5 minutes. Assuming that the amplitude decreases according to the exponential law, find the percentage rate of decrease of the amplitude. Find the amplitude at the end of 10 minutes.

94. Hyperbolic Functions. The hyperbolic functions now to be defined and discussed very briefly present many analogies with the circular functions. These functions are the hyperbolic sine, the hyperbolic cosine, the hyperbolic tangent, etc. They are written sinh u, cosh u, tanh u, etc., respectively, where u is the argument of the function.

The hyperbolic functions are defined by the equations:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \operatorname{sech} x = \frac{1}{\cosh x},$$

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{1}{\tanh x}.$$

cosh x and sech x are even functions, while the remaining four are odd functions.

Exercises

1. By making use of the definitions the student will show that the following identities hold. They are analogous to those satisfied by the circular functions.

$$\cosh^2 x - \sinh^2 x = 1.$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x.$$

2. Show by the use of the defining equations that:

$$\frac{d \cosh x}{dx} = \sinh x.$$

$$\frac{d \sinh x}{dx} = \cosh x.$$

$$\frac{d \tanh x}{dx} = \operatorname{sech}^2 x.$$

$$\frac{d \coth x}{dx} = -\operatorname{csch}^2 x.$$

$$\frac{d \operatorname{sech} x}{dx} = -\operatorname{sech} x \tanh x.$$

$$\frac{d \operatorname{csch} x}{dx} = -\operatorname{csch} x \coth x.$$

3. Sketch the curves $y = \cosh x$, $y = \sinh x$, and $y = \tanh x$.

95. Inverse Hyperbolic Functions. The logarithms of certain functions can be expressed in terms of inverse hyperbolic functions.

Let

$$y = \sinh^{-1} x.$$

$$x = \sinh y = \frac{e^y - e^{-y}}{2},$$

or

$$e^{2y} - 2xe^y - 1 = 0,$$

whence

$$e^y = x \pm \sqrt{x^2 + 1}.$$

The minus sign cannot be taken since e^y is always positive. Hence

$$e^y = x + \sqrt{x^2 + 1},$$

and

$$y = \sinh^{-1} x = \log (x + \sqrt{x^2 + 1}).$$

Exercises

1. Show that

$$\cosh^{-1} x = \log (x \pm \sqrt{x^2 - 1}).$$

Since

$$x - \sqrt{x^2 - 1} = \frac{1}{x + \sqrt{x^2 - 1}},$$

$$\log (x - \sqrt{x^2 - 1}) = -\log (x + \sqrt{x^2 - 1}).$$

Therefore

$$\cosh^{-1}x = \pm \log (x + \sqrt{x^2 - 1}).$$

The inverse hyperbolic cosine is then not single valued. Two values of $\cosh^{-1} x$, equal numerically but of opposite sign, correspond to each value of x greater than 1.

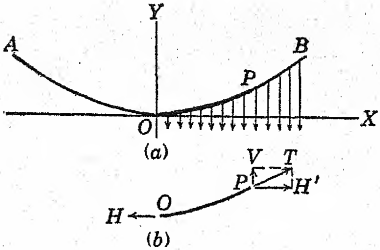

FIG. 64.

2. Show that:

$$\tanh^{-1} x = \tfrac{1}{2} \log \frac{1+x}{1-x}, \qquad \text{if } x^2 < 1;$$

$$\coth^{-1} x = \tfrac{1}{2} \log \frac{x+1}{x-1}, \qquad \text{if } x^2 > 1;$$

$$\operatorname{sech}^{-1} x = \pm\log \frac{1+\sqrt{1-x^2}}{x}, \qquad \text{if } 0 < x \leqq 1;$$

$$\operatorname{csch}^{-1} x = \log \frac{1+\sqrt{x^2+1}}{x}, \qquad \text{if } x > 0;$$

and

$$\operatorname{csch}^{-1} x = \log \frac{1-\sqrt{x^2+1}}{x}, \qquad \text{if } x < 0.$$

The student is not advised to memorize the formulas of this and the preceding section at this point in his course, but to acquire sufficient familiarity with the hyperbolic functions to enable him to operate with these functions by referring to the definitions and formulas given here and to others that he will find in mathematical tables.

96. The Catenary. Let AOB, Fig. 64 a, be a cable suspended from the points A and B and carrying only its weight. Let

us find the equation of the curve assumed by the cable. We shall assume that the curve has a vertical line of symmetry, OY, and that the tangent line drawn to the curve at O is horizontal.

Take OY as the Y-axis. Imagine a portion of the curve OP of length s, cut free. To hold this portion in equilibrium the forces H and T, Fig. 64*b*, must be introduced at the cut ends. H and T are, respectively, equal to the tension in the cable at the points O and P and they act in the direction of the tangent lines drawn to the curve at these points. The portion of the cable OP, Fig. 64*b*, is in equilibrium. Hence H', the horizontal component of T, is equal to H.

V, the vertical component of T, must balance the weight of the portion OP of the cable. Hence

$$V = sw,$$

where w is the weight of a unit length of the cable.

From Fig. 64*b*, it is seen that

$$\frac{dy}{dx} = \frac{V}{H'} = \frac{V}{H} = \frac{ws}{H}.$$

Let

$$\frac{w}{H} = \frac{1}{a}.$$

Then

$$\frac{dy}{dx} = \frac{s}{a}. \tag{1}$$

This differential equation involves three variables, *viz.*, x, y and s. s may be eliminated by differentiating and substituting for $\frac{ds}{dx}$ its value,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Thus

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{1}{a}\frac{ds}{dx} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

The equation now involves only two variables and may be written

$$\frac{d\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{1}{a}\,dx. \tag{2}$$

If we look upon $\frac{dy}{dx}$ as the variable u, the left-hand side of equation (2) is

$$\frac{du}{\sqrt{1+u^2}}$$

whose integral is $\log\,(u + \sqrt{1+u^2})$ [see formula (6), §89]. Integrating (2),

$$\log\left[\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}\right] = \frac{x}{a} + C. \tag{3}$$

When $x = 0$, $\frac{dy}{dx} = 0$.
Hence $C = 0$ and (3) becomes

$$\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = e^{\frac{x}{a}}. \tag{4}$$

From the symmetry of the curve $\frac{dy}{dx}$ changes sign when x is replaced by $-x$. Then from (4),

$$-\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = e^{-\frac{x}{a}}. \tag{5}$$

Subtracting (5) from (4),

$$2\frac{dy}{dx} = e^{\frac{x}{a}} - e^{-\frac{x}{a}}, \tag{6}$$

or

$$\frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) = \sinh\frac{x}{a}. \tag{7}$$

Integrating (7),

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) + C = a \cosh \frac{x}{a} + C. \tag{8}$$

If the origin is taken a units below the point O, Fig. 64a, $y = a$ when $x = 0$, and $C = 0$. Hence

$$y = a \cosh \frac{x}{a}. \tag{9}$$

This is the equation of the curve assumed by the cable. It is called the catenary.

Equation (9) can be written

$$Y = \cosh X, \tag{10}$$

where

$$Y = \frac{y}{a} \text{ and } X = \frac{x}{a}.$$

The constant a depends upon the tautness of the cable. Equation (10) shows that the curve $y = \cosh x$ if magnified the proper number of diameters will fit any cable hanging under its own weight

The length of OP can be found by substituting in formula (2), §64, the value of $\frac{dy}{dx}$ given by (7), and integrating.

$$ds = \sqrt{1 + \frac{\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)^2}{4}}\,dx = \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}dx.$$

Then

$$s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) + C_3.$$

Or using the hyperbolic functions,

$$ds = \sqrt{1 + \sinh^2 \frac{x}{a}}\, dx$$

$$= \cosh \frac{x}{a}\, dx.$$

$$s = a \sinh \frac{x}{a} + C_3.$$

Since s is measured from the point where the curve crosses the Y-axis, $s = 0$ when $x = 0$. Hence $C_3 = 0$ and

$$s = a \sinh \frac{x}{a}. \tag{11}$$

Exercises

1. From the relation $T = \sqrt{V^2 + H^2} = w\sqrt{s^2 + a^2}$, show that $T = wy$.

2. If the two supports A and B, Fig. 64, are at the same level, and L feet apart, and if the length of the cable is l feet, obtain an equation for determining the quantity a which appears in the equation of the catenary. If $l = 120$ and $L = 100$, find a by using a table of hyperbolic functions and the method of successive trials.

3. If the cable, Fig. 64, is drawn very taut, show that the equation of its curve is approximately

$$y = \frac{x^2}{2a},$$

if the origin of coordinates is taken at the lowest point of the cable.

HINT. Begin with equation (2) and note that $\left(\frac{dy}{dx}\right)^2$ is small compared with 1.

CHAPTER X

POLAR COORDINATES

97. Direction of Curve in Polar Coordinates. Let BPQ, Fig. 65, be a curve referred to O as pole and OA as polar axis. Let P be any point of the curve and let PT be a tangent to the curve at this point. Let PS lie in the radius vector OP, produced.

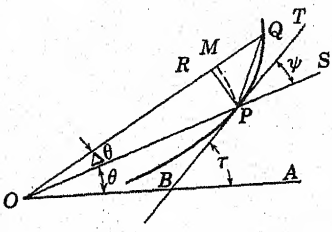

FIG. 65.

A point describing the curve, when at P, moves in the direction PT. This direction is given by the angle ψ through which the radius vector produced must rotate in a positive direction about P, in order to become coincident with the tangent line.

An expression for $\tan \psi$ will now be found. Let Q, Fig. 65, be a second point of the curve. PR is drawn perpendicular to OQ, and PM is a circular arc with O as center and radius $OP = \rho$.

$$\tan \psi = \lim_{\Delta\theta \to 0} \tan RQP = \lim_{\Delta\theta \to 0} \frac{PR}{RQ}. \tag{1}$$

The infinitesimals PR and RQ can be replaced by the infinitesimals $PM = \rho\Delta\theta$ and $MQ = \Delta\rho$, respectively, if (see §59)

$$\lim_{\Delta\theta \to 0} \frac{PR}{PM} = 1 \tag{2}$$

and

$$\lim_{\Delta\theta \to 0} \frac{RQ}{MQ} = 1. \tag{3}$$

Equation (2) is true by equation (3), §55. The proof of equation (3) follows:

$$\begin{aligned}\lim_{\Delta\theta\to 0}\frac{RQ}{MQ} &= \lim_{\Delta\theta\to 0}\frac{RM + MQ}{MQ}\\ &= \lim_{\Delta\theta\to 0}\frac{\rho(1-\cos\Delta\theta)+\Delta\rho}{\Delta\rho}\\ &= 1 + \lim_{\Delta\theta\to 0}\frac{\rho(1-\cos\Delta\theta)}{\Delta\theta}\,\frac{\Delta\theta}{\Delta\rho}.\end{aligned}$$

Hence

$$\lim_{\Delta\theta\to 0}\frac{RQ}{MQ} = 1,$$

since

$$\lim_{\Delta\theta\to 0}\frac{1-\cos\Delta\theta}{\Delta\theta} = 0.$$

From (1), (2), and (3) it follows that

$$\tan\psi = \lim_{\Delta\theta\to 0}\frac{PM}{MQ} = \lim_{\Delta\theta\to 0}\frac{\rho\Delta\theta}{\Delta\rho}.$$

Hence

$$\tan\psi = \rho\frac{d\theta}{d\rho}.$$

$$\tan\psi = \frac{\rho}{\dfrac{d\rho}{d\theta}}. \tag{4}$$

This formula can be recalled easily by drawing a figure like Fig. 65 and writing

$$\tan\psi = \lim_{\Delta\theta\to 0}\frac{PR}{RQ} = \lim_{\Delta\theta\to 0}\frac{PM}{MQ} = \lim_{\Delta\theta\to 0}\frac{\rho\Delta\theta}{\Delta\rho} = \rho\frac{d\theta}{d\rho}.$$

Illustration 1. If $\rho = e^{a\theta}$,

$$\frac{d\rho}{d\theta} = ae^{a\theta}$$

and $\tan\psi = \frac{1}{a}$, a constant.

Illustration 2. Find the equation of the family of curves for which the angle between the radius vector produced and the tangent line is a constant.

$$\tan \psi = k.$$

$$\frac{\rho}{\frac{d\rho}{d\theta}} = k,$$

or

$$\frac{\frac{d\rho}{d\theta}}{\rho} = \frac{1}{k}.$$

$$\frac{d\rho}{\rho} = \frac{1}{k}d\theta.$$

Integrating,

$$\log \rho = \frac{\theta}{k} + C$$

$$\rho = e^{\frac{\theta}{k}+C}$$

$$= e^{C} e^{\frac{\theta}{k}}.$$

or

$$\rho = Ke^{\frac{\theta}{k}},$$

where K is an arbitrary constant.

Exercises

Find $\tan \psi$ for each of the following twelve curves:

1. $\rho = a\theta.$
2. $\rho = \frac{a}{\theta}.$
3. $\rho = a \sin 2\theta.$
4. $\rho = a \sin 3\theta.$
5. $\rho = 1 + \tan \theta.$
6. $\rho = a^{\theta}.$
7. $\rho = a \cos \theta.$
8. $\rho = a(1 - \cos \theta).$
9. $\rho = \frac{a}{1 - \cos \theta}.$
10. $\rho = a \cos (\theta - \alpha).$
11. $\rho^2 = a^2 \cos 2\theta.$
12. $\rho = \frac{a}{1 + \sin \theta}.$

13. Find the polar equation of the curve passing through $\left(4, \frac{\pi}{4}\right)$, if $\psi = \theta$ for every point of the curve. Sketch the curve.

14. Find the polar equation of the curve passing through (2,0) if ψ for every point of the curve is a constant α which is not equal to $\frac{\pi}{2} \pm n\pi$.

15. Find the polar equation of the curve passing through $\left(1, \frac{\pi}{4}\right)$, if $\psi = \frac{\pi}{2} - \theta$ for every point of the curve. Sketch the curve.

98. Differential of Arc: Polar Coordinates. We shall now find an expression for $\frac{ds}{d\theta}$ in polar coordinates. From Fig. 65,

$$(\text{chord } PQ)^2 = (PR)^2 + (RQ)^2.$$

From which

$$\lim_{\Delta\theta\to 0}\left(\frac{\text{chord } PQ}{\Delta\theta}\right)^2 = \lim_{\Delta\theta\to 0}\left(\frac{PR}{\Delta\theta}\right)^2 + \lim_{\Delta\theta\to 0}\left(\frac{RQ}{\Delta\theta}\right)^2.$$

Replacing chord PQ by arc $PQ = \Delta s$, PR by $PM = \rho\,\Delta\theta$, and RQ by $MQ = \Delta\rho$,

$$\lim_{\Delta\theta\to 0}\left(\frac{\Delta s}{\Delta\theta}\right)^2 = \lim_{\Delta\theta\to 0}\left(\frac{\rho\Delta\theta}{\Delta\theta}\right)^2 + \lim_{\Delta\theta\to 0}\left(\frac{\Delta\rho}{\Delta\theta}\right)^2.$$

Therefore

$$\left(\frac{ds}{d\theta}\right)^2 = \rho^2 + \left(\frac{d\rho}{d\theta}\right)^2 \tag{1}$$

and

$$ds = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\,d\theta. \tag{2}$$

This formula can be written

$$(ds)^2 = \rho^2(d\theta)^2 + (d\rho)^2. \tag{3}$$

It can be recalled easily by the aid of the triangle PRQ of Fig. 65.

From this triangle it follows that

$$\sin\psi = \lim_{\Delta\theta\to 0} \sin RQP = \lim_{\Delta\theta\to 0} \frac{PR}{PQ} = \rho\frac{d\theta}{ds}$$

$$\cos\psi = \lim_{\Delta\theta\to 0} \cos RQP = \lim_{\Delta\theta\to 0} \frac{RQ}{PQ} = \frac{d\rho}{ds}.$$

From (3) it follows that the velocity of a point moving in the curve $\rho = f(\theta)$ can be expressed in terms of ρ, $\frac{d\rho}{dt}$ and $\frac{d\theta}{dt}$ by

$$\frac{ds}{dt} = \sqrt{\rho^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\rho}{dt}\right)^2}. \tag{4}$$

The length of the curve can be expressed as a definite integral. Thus (see Fig. 65),

$$s = \lim_{\Delta\theta\to 0} \sum_{\theta=\alpha}^{\theta=\beta} PQ$$

$$= \lim_{\Delta\theta\to 0} \sum_{\theta=\alpha}^{\theta=\beta} \sqrt{(PR)^2 + (RQ^2)}$$

$$= \lim_{\Delta\theta\to 0} \sum_{\theta=\alpha}^{\theta=\beta} \sqrt{\left(\frac{PR}{\Delta\theta}\right)^2 + \left(\frac{RQ}{\Delta\theta}\right)^2}\Delta\theta.$$

Noting that

$$\lim_{\Delta\theta\to 0} \frac{PR}{\Delta\theta} = \rho.$$

and that

$$\lim_{\Delta\theta\to 0} \frac{RQ}{\Delta\theta} = \frac{d\rho}{d\theta}$$

and applying Duhamel's theorem, it follows that

$$s = \lim_{\Delta\theta\to 0} \sum_{\theta=\alpha}^{\theta=\beta} \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\,\Delta\theta$$

$$= \int_\alpha^\beta \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\,d\theta.$$

Illustration. Find the entire length of the curve $\rho = a(1 - \cos\theta)$. The length of the upper half will be found and multiplied by 2.

$$\frac{d\rho}{d\theta} = a \sin\theta.$$

$$\begin{aligned} \frac{s}{2} &= \int_0^\pi \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta \\ &= \int^\pi \sqrt{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta}\, d\theta \\ &= 2a \int_0^\pi \sqrt{\frac{1 - \cos\theta}{2}}\, d\theta \\ &= 2a \int_0^\pi \sin\frac{\theta}{2}\, d\theta \\ &= -4a \cos\frac{\theta}{2}\Big|_0^\pi = 4a. \end{aligned}$$

$$s = 8a.$$

Exercises

1. Find the entire length of the curve $\rho = 2a \sin\theta$.

2. Find the entire length of $\rho = a\cos^2\frac{\theta}{2}$.

3. Find the entire length of the curve $\rho = a \sin^3\frac{\theta}{3}$.

4. Find the length of $\rho = e^{a\theta}$ between the points corresponding to $\theta = 0$ and $\theta = \pi$; also between the points corresponding to $\theta = 0$ and $\theta = \frac{\pi}{3}$.

5. Prove formula (3) directly from

$$x = \rho \cos\theta,$$
$$y = \rho \sin\theta,$$

and

$$ds = \sqrt{(dx)^2 + (dy)^2}.$$

6. Find the length of that part of $\rho = 4 \cos\theta$ which lies outside of $\rho = 2$.

7. A particle moves on the curve $\rho = 5(2 + \cos\theta)$, the unit of length being 1 foot, in such a way that θ increases uniformly at the rate of 1.5 radians per second. Find the velocity in magnitude and direction when $\theta = 60°$; when $\theta = 90°$; when $\theta = 180°$.

8. A cam, whose boundary, referred to the pole O and the axis of symmetry OP, Fig. 66, is

$$\rho = 6 + 3\cos\theta,$$

the unit of length being 1 inch, revolves uniformly about O, making 30 revolutions a minute. The follower MN slides in vertical guides and if extended would pass through O. Find the velocity of the follower when the axis of symmetry OP makes an angle of 60° with the horizontal.

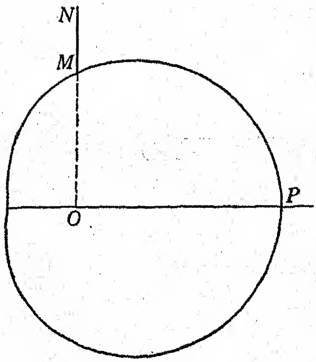

FIG. 66.

99. Area: Polar Coordinates. Find the area bounded by the curve $\rho = f(\theta)$ and the radii $\theta = \alpha$ and $\theta = \beta$. We seek the area

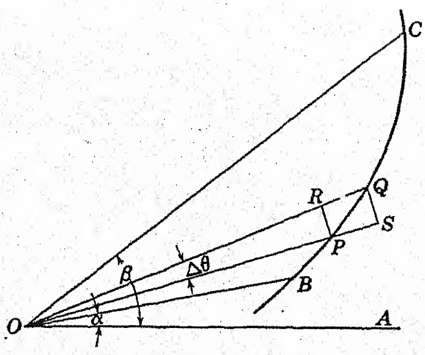

FIG. 67.

BOC, Fig. 67. Draw radii dividing the angle BOC into n equal parts $\Delta\theta$. Let POQ be a typical one of the n portions into which the area is divided by these radii. The angle POQ is $\Delta\theta$.

The line OP makes an angle θ with the initial line OA, and its length is $\rho = f(\theta)$. Denote the area of BOC by A.

$$A = \lim_{\Delta\theta \to 0} \sum POQ. \tag{1}$$

In accordance with Duhamel's theorem[1] we may replace each sector POQ by a sector POR whose area is $\frac{1}{2}\rho^2\Delta\theta$. Then

$$A = \lim_{\Delta\theta \to 0} \sum_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2}\rho^2\,\Delta\theta,$$

$$A = \tfrac{1}{2}\int_\alpha^\beta \rho^2\,d\theta,$$

or

$$A = \tfrac{1}{2}\int_\alpha^\beta [f(\theta)]^2 d\theta. \tag{2}$$

Exercises

Find the area bounded by

1. One loop of $\rho = 10 \cos 2\theta$.
2. One loop of $\rho = 5 \sin 2\theta$.
3. One loop of $\rho = 10 \sin 3\theta$.
4. One loop of $\rho = 10 \cos n\theta$.
5. $\rho = 2a(1 + \cos\theta)$.
6. $\rho = 3 + 2\cos\theta$.
7. The smaller loop of $\rho = 1 + 2\cos\theta$.
8. The smaller loop of $\rho = \sqrt{3} + 2\sin\theta$.
9. The smaller loop of $\rho = 10\cos\dfrac{\theta}{2}$.

[1] In Fig. 67, let $\Delta A = OPQ$. PR and QS are arcs of circles. Then

$$OPR < \Delta A < OSQ,$$

i.e.,

$$\tfrac{1}{2}\rho^2\,\Delta\theta < OPQ < \tfrac{1}{2}(\rho + \Delta\rho)^2\Delta\theta$$

It follows that

$$\lim_{\Delta\theta \to 0} \frac{OPQ}{\frac{1}{2}\rho^2\,\Delta\theta} = 1.$$

10. $\rho = \theta$ and the radii $\theta = 0$ and $\theta = \frac{\pi}{2}$.

11. $\rho = \frac{10}{\theta^2}$ and the radii $\theta = \frac{\pi}{4}$ and $\theta = \pi$.

12. $\rho = 5\theta^2$ and the radii $\theta = 0$ and $\theta = \frac{\pi}{2}$.

13. Find the area outside $\rho = 2$ and inside $\rho = 4 \cos \theta$.

14. Find the area outside $\rho = 6$ and inside $\rho = 4(1 + \cos \theta)$.

15. Find the area outside of $\rho = 3 + \sin \theta$ and inside of $\rho = 2 + 3 \sin \theta$.

CHAPTER XI
INTEGRATION

100. Formulas. In Chapters III, VI, and VII the following formulas of integration, with the exception of (19), have been used. They are collected here for reference, and should be memorized by the student.

1. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$, if $n \neq -1$.

2. $\int \frac{du}{u} = \log u + C.$

3. $\int e^u\,du = e^u + C.$

4. $\int a^u\,du = \frac{1}{\log_e a}a^u + C.$

5. $\int \sin u\,du = -\cos u + C.$

6. $\int \cos u\,du = \sin u + C.$

7. $\int \sec^2 u\,du = \tan u + C.$

8. $\int \csc^2 u\,du = -\cot u + C.$

9. $\int \sec u \tan u\,du = \sec u + C.$

10. $\int \csc u \cot u\,du = -\csc u + C.$

11. $\int \tan u\,du = \log \sec u + C.$

12. $\int \cot u \, du = \log \sin u + C.$

13. $\int \sec u \, du = \log (\sec u + \tan u) + C.$

14. $\int \csc u \, du = -\log (\csc u + \cot u) + C.$

15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C.$

16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C.$

17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C.$

18. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log (u + \sqrt{u^2 \pm a^2}) + C.$

19. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\log\frac{u - a}{u + a} + C, \text{ if } u > a$

$$= \frac{1}{2a}\log\frac{a - u}{a + u} + C, \text{ if } u < a.$$

Formula (19) is proved as follows:

$$\frac{1}{u^2 - a^2} = \frac{1}{2a}\left[\frac{1}{u - a} - \frac{1}{u + a}\right]$$

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\int\left[\frac{1}{u - a} - \frac{1}{u + a}\right]du$$

$$= \frac{1}{2a}\int\frac{du}{u - a} - \frac{1}{2a}\int\frac{du}{u + a}$$

$$= \frac{1}{2a}\log (u - a) - \frac{1}{2a}\log (u + a) + C$$

$$= \frac{1}{2a}\log\frac{u - a}{u + a} + C.$$

This formula leads to the logarithm of a negative number if $u < a$. To obtain a formula for this case write

$$\frac{1}{u^2 - a^2} = \frac{1}{2a}\left[-\frac{1}{a - u} - \frac{1}{a + u}\right].$$

Then

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \frac{a - u}{a + u} + C.$$

Exercises

1. $\int \frac{dx}{x^2 - 16}$.

2. $\int \frac{dx}{\sqrt{x^2 - 25}}$.

3. $\int \frac{dx}{x^2 + 36}$.

4. $\int \frac{dx}{x^2 - 4}$.

5. $\int \frac{dx}{9 - x^2}$.

6. $\int \frac{x\,dx}{x^2 - 25}$.

7. $\int \frac{dx}{x\sqrt{x^2 - 16}}$.

8. $\int \cot 7t\,dt$.

9. $\int \frac{(x + a)\,dx}{x^2 + 2ax}$.

10. $\int (x^2 - 16)^{\frac{3}{2}} x\,dx$.

11. $\int \sin (2x - 3)\,dx$.

12. $\int \sec^2 (5\alpha + 2)\,d\alpha$.

13. $\int \sec (2\theta + 4) \tan (2\theta + 4) d\theta$.

14. $\int \csc^2 (3 - 2\phi)\,d\phi$.

15. $\int e^{-x^2} x\,dx$.

16. $\int e^{\cos\theta} \sin \theta\,d\theta$.

17. $\int \frac{3t^2\,dt}{30t^3 + 13}$.

18. $\int (\sqrt{c} - \sqrt{x})^3 dx$.

19. $\int \sqrt{3 + 4x}\,dx$.

20. $\int \frac{dy}{\sqrt{5 - 3y}}$.

21. $\int e^{\frac{y}{3}}\,dy$.

22. $\int e^{\tan(2x+3)} \sec^2 (2x + 3)\,dx$.

23. $\int \frac{x}{x + 1}\,dx = \int \left(1 - \frac{1}{x + 1}\right) dx = \int dx - \int \frac{dx}{x + 1}$.

24. $\int \frac{x^2 + 2}{x + 1}\,dx$. Divide numerator by denominator.

25. $\int \left[e^{\frac{x}{2}} + e^{-\frac{x}{2}} \right] dx.$

26. $\int a^{5x}\, dx.$

27. $\int (2x + 4)^{0.52}\, dx.$

28. $\int \frac{t^3 + 3}{t + 1}\, dt.$

29. $\int \frac{dx}{7x^2 + 11}.$

30. $\int \frac{dx}{\cos^2 (3x - 2)}.$

31. $\int e^{3x}\, dx.$

32. $\int \frac{dx}{4x^2 - 9}.$

33. $\int \frac{\cos x\, dx}{4 + 3 \sin x}.$

34. $\int \frac{dx}{\sqrt{16 - 9x^2}}.$

35. $\int \tan (3\alpha + 4)\, d\alpha.$

36. $\int (\tan \theta + \cot \theta)^2\, d\theta = \tan \theta - \cot \theta + C.$

37. $\int \left(\sin \frac{\theta}{5} - \cos 5\theta\right) d\theta.$

38. $\int \cot (5t - 8)\, dt.$

39. $\int \cos (3t - 4)\, dt.$

40. $\int \frac{y\, dy}{9 - 4y^2}.$

41. $\int \frac{dz}{\sqrt{3z + 7}}.$

42. $\int \frac{dx}{7 - 5x^2}.$

43. $\int \tan (2x - 5)\, dx.$

44. $\int \sec (2y + 4)\, dy.$

45. $\int \csc (2y - 7)\, dy.$

46. $\int \cot (3t + 11)\, dt.$

47. $\int \sec^2 \left(\frac{x}{3} - 5\right) dx.$

48. $\int \cos (3 - 2x)\, dx.$

49. $\int \frac{2x + 5}{x^2 + 5x + 41}\, dx.$

50. $\int \frac{dt}{9t^2 + 4}.$

51. $\int \frac{dt}{4t^2 - 25}.$

52. $\int \frac{dt}{\sqrt{9t^2 - 16}}.$

53. $\int \frac{dt}{\sqrt{16 - 25t^2}}$.

54. $\int \frac{dt}{t\sqrt{9t^2 - 4}}$.

55. $\int \frac{t\,dt}{\sqrt{25t^2 - 9}}$.

56. $\int \frac{t\,dt}{4t^2 - 25}$.

57. $\int \sec 5x\,dx$.

58. $\int \sin(\omega t + \alpha)\,dt$.

59. $\int \cos^2 4x \sin 4x\,dx$.

60. $\int \sin^4(x+3)\cos(x+3)\,dx$.

61. $\int \cos^5(3x-2)\sin(3x-2)dx$.

62. $\int \sec^2(9 - 7x)\,dx$.

63. $\int e^{\tan\frac{x}{3}} \sec^2 \frac{x}{3}\,dx$.

64. $\int \tan^4 x \sec^2 x\,dx$.

65. $\int \cos^2 3x \sin 3x\,dx$.

66. $\int \tan^3 5x \sec^2 5x\,dx$.

67. $\int \sec^4 x \tan x\,dx$.

68. $\int \csc^5 x \cot x\,dx$.

69. $\int x \tan(2x^2 - 5)\,dx$.

70. $\int \frac{\sin x \cos x\,dx}{4 + \sin^2 x}$.

71. $\int \frac{\cos x\,dx}{4 + \sin^2 x}$.

72. $\int e^{\frac{1}{x}} \frac{1}{x^2}\,dx$.

73. $\int (a^{\frac{2}{3}} - x^{\frac{2}{3}})^3\,dx$.

74. $\int \frac{dt}{4 - 25t^2}$.

75. $\int t\sqrt{6t^2 - 17}dt$.

76. $\int \frac{x\,dx}{\sqrt{9 - x^2}}$.

77. $\int \frac{\sin 5x\,dx}{3\cos 5x + 11}$.

78. $\int e^{\sin 3x} \cos 3x\,dx$.

79. $\int e^{x^2+6x+7}(x+3)\,dx$.

80. $\int \cos 5x \sin 3x\,dx$.

81. $\int \cos 3x \cos 5x\,dx$.

82. $\int \sin 7x \sin 4x\,dx$.

83. $\int \sin 3t \cos 4t\,dt$.

84. $\int_0^{\pi} \sin^2 5t\,dt$.

85. $\int_0^{\pi} \sin mt \sin nt\, dt$, where m and n are integers. What is the value if $m = n$?

86. $\int \cos (3\omega t + \alpha) \sin (3\omega t + \alpha) dt.$

87. $\int \frac{2x + 3}{x^2 + 9}\, dx.$

88. $\int \frac{x^3 + 3x^2 + 7}{x^2 + 9}\, dx.$

89. $\int \frac{x + 1}{3x^2 - 11}\, dx.$

90. $\int \frac{3x + 2}{4x^2 - 16}\, dx.$

91. $\int \sin^3 5\theta \cos 5\theta\, d\theta.$

92. $\int \sin^2 x\, dx.$

93. $\int x\sqrt{16 - x^2}\, dx$

94. $\int \sec^2\left(\frac{x}{3} + 2\right)\, dx.$

95. $\int \frac{t^2}{5t^3 + 7}\, dt.$

96. $\int \sec^3 4x \tan 4x\, dx.$

97. $\int \frac{e^{\sqrt{x+2}}}{\sqrt{x + 2}}\, dx.$

98. $\int \frac{dx}{\sqrt{5 - 7x^2}}.$

99. $\int \frac{dx}{\sqrt{3x^2 - 5}}.$

100. $\int \sin 4x \cos 6x\, dx.$

101. $\int (\sqrt{a} - \sqrt{x})^2\, dx.$

102. $\int \frac{2x + 3}{2x + 7}\, dx.$

103. $\int \frac{dx}{\sqrt{3x + 2}}.$

104. $\int \sin^4 2x \cos 2x\, dx.$

105. $\int \sqrt{\sin x} \cos x\, dx.$

106. $\int e^{-3t}\, dt.$

107. $\int (2x - 5)^{\frac{1}{3}}\, dx.$

108. $\int \sec \frac{x}{2}\, dx.$

109. $\int \sec (3\phi - 2) \tan (3\phi - 2)\, d\phi.$

110. $\int \tan^5 (2x - 1) \sec^2 (2x - 1)\, dx.$

111. $\displaystyle\int \frac{\sec^2 3x}{1 + \tan 3x}\,dx.$

112. $\displaystyle\int \sqrt{2 - 3x}\,dx.$

113. $\displaystyle\int \tan\,(5 - 2x)dx.$

114. $\displaystyle\int \frac{x\,dx}{5 - 3x^2}.$

115. $\displaystyle\int \frac{dy}{y^2 - 7}.$

116. $\displaystyle\int \frac{x + 2}{x + 5}\,dx.$

117. $\displaystyle\int \frac{x + 4}{x^2 - 9}\,dx.$

118. $\displaystyle\int \frac{x\,dx}{3x^2 + 4}.$

119. $\displaystyle\int \frac{dy}{y\sqrt{3y^2 - 7}}.$

120. $\displaystyle\int \frac{y\,dy}{\sqrt{3y^2 - 7}}.$

121. $\displaystyle\int \sec^5\theta \tan\theta\,d\theta.$

122. $\displaystyle\int \frac{dy}{\sqrt{5y^2 - 13}}.$

123. $\displaystyle\int \sin^3 \frac{x}{3} \cos \frac{x}{3}\,dx.$

124. $\displaystyle\int \cos 2x \sin \frac{x}{2}\,dx.$

101. Integration of Expressions Containing $ax^2 + bx + c$, by Completing the Square.

Illustration 1.

$$\int \frac{dx}{x^2 + 4x + 9} = \int \frac{dx}{(x + 2)^2 + 5} = \frac{1}{\sqrt{5}} \tan^{-1} \frac{x + 2}{\sqrt{5}} + C.$$

Illustration 2.

$$\int \frac{dx}{\sqrt{3 + 4x - x^2}} = \int \frac{dx}{\sqrt{3 - (x^2 - 4x)}} = \int \frac{dx}{\sqrt{3 + 4 - (x - 2)^2}}$$

$$= \int \frac{dx}{\sqrt{7 - (x - 2)^2}} = \sin^{-1} \frac{x - 2}{\sqrt{7}} + C.$$

Illustration 3.

$$\int \frac{dx}{7x^2 + 3x + 11} = \frac{1}{7}\int \frac{dx}{x^2 + \frac{3}{7}x + \frac{9}{196} + \frac{11}{7} - \frac{9}{196}}$$

$$= \frac{1}{7}\int \frac{dx}{(x + \frac{3}{14})^2 + \frac{299}{196}}$$

$$= \frac{1}{7}\,\frac{14}{\sqrt{299}}\tan^{-1}\frac{x+\frac{3}{14}}{\frac{\sqrt{299}}{14}} + C$$

$$= \frac{2}{\sqrt{299}}\tan^{-1}\frac{14x+3}{\sqrt{299}} + C.$$

Illustration 4.

$$\int\frac{dx}{\sqrt{6+2x-3x^2}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{2+\frac{2}{3}x-x^2}}$$

$$= \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{2-(x^2-\frac{2}{3}x)}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\frac{19}{9}-(x-\frac{1}{3})^2}}$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\frac{x-\frac{1}{3}}{\frac{1}{3}\sqrt{19}} + C = \frac{1}{\sqrt{3}}\sin^{-1}\frac{3x-1}{\sqrt{19}} + C.$$

Exercises

1. $\displaystyle\int\frac{dx}{x^2+4x+20}.$

2. $\displaystyle\int\frac{dx}{\sqrt{7+6x-x^2}}.$

3. $\displaystyle\int\frac{dx}{\sqrt{x^2+4x+11}}.$

4. $\displaystyle\int\frac{dx}{\sqrt{6x-x^2}}.$

5. $\displaystyle\int\frac{dx}{x^2+6x-25}.$

6. $\displaystyle\int\frac{dx}{2x^2+12x+50}.$

7. $\displaystyle\int\frac{dx}{\sqrt{3x^2+6x+12}}.$

8. $\displaystyle\int\frac{dx}{\sqrt{16+4x-2x^2}}.$

9. $\displaystyle\int\frac{dx}{3x^2+4x+6}.$

10. $\displaystyle\int\frac{dx}{2x^2+8x-10}.$

11. $\displaystyle\int\frac{dx}{\sqrt{6-4x-2x^2}}.$

12. $\displaystyle\int\frac{dx}{2x^2+14x+25}.$

13. $\displaystyle\int\frac{dx}{\sqrt{5-3x-2x^2}}.$

14. $\displaystyle\int\frac{dx}{\sqrt{10+4x-3x^2}}.$

15. $\displaystyle\int \frac{dx}{3x^2 - 8x + 4}.$

16. $\displaystyle\int \frac{dx}{x\sqrt{2x^2 + 3x - 2}}.$ Substitute $x = \dfrac{1}{z}.$

17. $\displaystyle\int \frac{dx}{x\sqrt{3 + 6x + 5x^2}}.$

102. Integrals Containing Fractional Powers of x or of a + bx. *Illustration* 1.

$$\int \frac{(x + 2)\,dx}{x\sqrt{2x + 3}}.$$

Let

$$2x + 3 = z^2.$$

Then

$$x = \frac{z^2 - 3}{2}.$$

$$dx = z\,dz$$

$$x + 2 = \frac{z^2 + 1}{2}.$$

$$\int \frac{(x + 2)dx}{x\sqrt{2x + 3}} = \int \frac{\dfrac{z^2 + 1}{2}\,z\,dz}{z\,\dfrac{(z^2 - 3)}{2}} = \int \frac{z^2 + 1}{z^2 - 3}\,dz$$

$$= \int \left(1 + \frac{4}{z^2 - 3}\right) dz$$

$$= z + \frac{4}{2\sqrt{3}} \log \frac{z - \sqrt{3}}{z + \sqrt{3}} + C$$

$$= \sqrt{2x + 3} + \frac{2\sqrt{3}}{3} \log \frac{\sqrt{2x + 3} - \sqrt{3}}{\sqrt{2x + 3} + \sqrt{3}} + C.$$

Illustration 2.

$$\int \frac{dx}{x^{\frac{2}{3}} + 4}.$$

Let

$$x = z^3.$$

$$\int \frac{dx}{x^{\frac{2}{3}} + 4} = 3\int \frac{z^2\,dz}{z^2 + 4}$$

$$= 3\int \left(1 - \frac{4}{z^2 + 4}\right) dz$$

$$= 3\left(z - 2\tan^{-1}\frac{z}{2}\right) + C$$

$$= 3\left(x^{\frac{1}{3}} - 2\tan^{-1}\frac{x^{\frac{1}{3}}}{2}\right) + C.$$

The method which has been used in the preceding illustrations applies when fractional powers of a single linear expression, $ax + b$, occur under the integral sign. The integral is simplified by the substitution

$$a + bx = z^n,$$

where n is so chosen that all fractional exponents disappear.

Exercises

1. $\displaystyle\int \frac{x + 3}{x\sqrt{x + 2}}\,dx.$

2. $\displaystyle\int \frac{3x - 2}{x\sqrt{x - 4}}\,dx.$

3. $\displaystyle\int \frac{dx}{x^{\frac{2}{3}} + x^{\frac{1}{2}}}.$ Let $x = z^6$.

4. $\displaystyle\int \frac{2x - 5}{x\sqrt{x - 1}}\,dx.$

5. $\displaystyle\int \frac{3x + 2}{x\sqrt{x + 1}}\,dx.$

6. $\displaystyle\int \frac{dx}{x\sqrt{3x - 2}}.$

7. $\displaystyle\int \frac{\sqrt{3x + 2}}{2x + 5}\,dx.$

8. $\displaystyle\int \frac{dx}{x^{\frac{3}{5}} + x^{\frac{1}{2}}}.$

9. $\displaystyle\int x\sqrt{x + 2}\,dx.$

10. $\displaystyle\int \frac{\sqrt{x + 1} + 1}{x}\,dx.$

11. $\displaystyle\int \frac{x\,dx}{(x+2)\sqrt{x-7}}$.

12. $\displaystyle\int \frac{x\,dx}{\sqrt[4]{2x+3}}$.

13. $\displaystyle\int \frac{dx}{x+\sqrt{2x-3}}$.

14. $\displaystyle\int \frac{dx}{x^{\frac{2}{3}}+3x^{\frac{3}{4}}}$.

15. $\displaystyle\int \frac{\sqrt{x-4}}{x+5}\,dx$.

16. $\displaystyle\int \frac{dx}{1+\sqrt{2x+3}}$.

103. Integrals of Powers of Trigonometric Functions.

(*a*) $\int \sin^m x \cos^n x\,dx$ where at least one of the exponents is an odd positive integer. This includes $\int \sin^m x\,dx$ and $\int \cos^n x\,dx$ where the exponents are odd.

Illustration 1.

$$\begin{aligned}\int \sin^3 x \cos^2 x\,dx &= \int (1-\cos^2 x)\cos^2 x \sin x\,dx\\ &= \int \cos^2 x \sin x\,dx - \int \cos^4 x \sin x\,dx\\ &= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.\end{aligned}$$

Illustration 2.

$$\begin{aligned}\int \cos^3 x\,dx &= \int (1-\sin^2 x)\cos x\,dx\\ &= \int \cos x\,dx - \int \sin^2 x \cos x\,dx\\ &= \sin x - \frac{\sin^3 x}{3} + C.\end{aligned}$$

It is seen that the process consists in combining one of the functions $\sin x$ or $\cos x$ with dx to form the differential of $\cos x$ or

of $\sin x$, respectively, and of expressing the remaining factors of the function to be integrated in terms of $\cos x$ or $\sin x$, respectively.

Exercises

1. $\int \sin^3 x\, dx.$
2. $\int \sin^5 x \cos^2 x\, dx.$
3. $\int \cos^3 2x\, dx.$
4. $\int \sin^4 x \cos^3 x\, dx.$
5. $\int \sin^5 x\, dx.$
6. $\int \sin^3 3x \cos^2 3x\, dx.$
7. $\int \sqrt{\cos x} \sin^3 x\, dx.$
8. $\int \sin^3 2x\, dx.$
9. $\int \frac{\sin^3 x}{\sqrt{\cos x}}\, dx.$
10. $\int \sin^3 x \cos^3 x\, dx.$
11. $\int \frac{\cos^5 \theta}{(\sin \theta)^{\frac{1}{3}}}\, d\theta.$
12. $\int \sin^2 (2x + 1) \cos^3 (2x + 1)\, dx.$
13. $\int \cos^2 (x + 3) \sin^3 (x + 3)\, dx.$
14. $\int \cos^5 2x\, dx.$

(*b*) $\int \sin^m x \cos^n x\, dx$ when m and n are both even positive integers. In this case make use of the relations:

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x).$$
$$\sin x \cos x = \tfrac{1}{2} \sin 2x.$$

Illustration 1.

$$\int \sin^2 x \cos^2 x\, dx = \tfrac{1}{4}\int \sin^2 2x\, dx = \tfrac{1}{8}\int (1 - \cos 4x)\, dx$$
$$= \frac{x}{8} - \frac{\sin 4x}{32} + C.$$

Illustration 2.

$$\int \cos^4 x\,dx = \tfrac{1}{4}\int (1 + \cos 2x)^2\,dx$$
$$= \tfrac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$
$$= \tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}\int (1 + \cos 4x)\,dx$$
$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C.$$

Illustration 3.

$$\int \sin^2 x \cos^4 x\,dx = \int (\sin x \cos x)^2 \cos^2 x\,dx$$
$$= \tfrac{1}{8}\int \sin^2 2x\,(1 + \cos 2x)\,dx$$
$$= \tfrac{1}{16}\int (1 - \cos 4x)dx + \tfrac{1}{8}\int \sin^2 2x \cos 2x\,dx$$
$$= \tfrac{1}{16}x - \tfrac{1}{64}\sin 4x + \tfrac{1}{48}\sin^3 2x + C.$$

Exercises

1. $\int \sin^2 x\,dx.$

2. $\int \sin^4 x\,dx.$

3. $\int \cos^4 2x\,dx.$

4. $\int \sin^2 3x\,dx.$

5. $\int \cos^4 5x\,dx.$

6. $\int \sin^6 x\,dx.$

7. $\int \sin^4 3x\,dx.$

8. $\int \cos^2 3x\,dx.$

(c) $\int \tan^n x\,dx$ and $\int \cot^n x\,dx.$

Illustration 1.

$$\int \tan^4 x\, dx = \int \tan^2 x\,(\sec^2 x - 1)\, dx = \frac{\tan^3 x}{3} - \int \tan^2 x\, dx$$
$$= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1)\, dx$$
$$= \frac{\tan^3 x}{3} - \tan x + x + C.$$

Illustration 2.

$$\int \cot^5 x\, dx = \int (\csc^2 x - 1)^2 \cot x\, dx$$
$$= \int \csc^4 x \cot x\, dx - 2\int \csc^2 x \cot x\, dx + \int \cot x\, dx$$
$$= -\tfrac{1}{4}\csc^4 x + \csc^2 x + \log \sin x + C.$$

(*d*) $\int \sec^n x\, dx$ and $\int \csc^n x\, dx$, n an even integer.

Illustration 1.

$$\int \sec^4 x\, dx = \int (1 + \tan^2 x)\sec^2 x\, dx = \tan x + \tfrac{1}{3}\tan^3 x + C.$$

When n is odd, this method fails (see §106).

(*e*) $\int \tan^m x \sec^n x\, dx$ and $\int \cot^m x \csc^n x\, dx$.

The methods illustrated below fail in certain cases. In such cases the method of §106 can be used.

Illustration 1.

$$\int \tan^4 x \sec^4 x\, dx = \int \tan^4 x(1 + \tan^2 x)\sec^2 x\, dx$$
$$= \tfrac{1}{5}\tan^5 x + \tfrac{1}{7}\tan^7 x + C.$$

Illustration 2.

$$\begin{aligned}\int \tan^3 x \sec^3 x\,dx &= \int \tan^2 x \sec^2 x \sec x \tan x\,dx\\ &= \int (\sec^2 x - 1)\sec^2 x \sec x \tan x\,dx\\ &= \int (\sec^4 x - \sec^2 x)\sec x \tan x\,dx\\ &= \tfrac{1}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C.\end{aligned}$$

Exercises

1. $\int \csc^4 x\,dx.$
2. $\int \cot^4 x\,dx.$
3. $\int \tan^2 x \sec^4 x\,dx.$
4. $\int \tan^4 2x\,dx.$
5. $\int \cot^3 x \csc^3 x\,dx.$
6. $\int \sec^6 x\,dx.$
7. $\int \tan^2 x \sec^2 x\,dx.$
8. $\int \frac{\sec^4 x}{\tan^2 x}\,dx.$
9. $\int \csc^6 x\,dx.$
10. $\int \cot^3 x \csc^5 x\,dx.$
11. $\int \tan^5 x \sec^3 x\,dx.$
12. $\int \cot^2 x\,dx.$
13. $\int \sqrt{\tan x}\,\sec^4 x\,dx.$
14. $\int \cot^2 \theta \csc^4 \theta\,d\theta.$
15. $\int \tan^4 2x \sec^2 2x\,dx.$
16. $\int \tan^3 x \sec^{\frac{3}{2}} x\,dx.$
17. $\int (\tan^2 x + \tan^4 x)\,dx.$
18. $\int \sec^4 3\theta\,d\theta.$

104. Integration of Expressions Containing $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$ by Trigonometric Substitution. The methods of §103 find frequent application in the integration of

expressions which result from the substitution of a trigonometric function for x in integrals containing radicals reducible to one of the forms $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, or $\sqrt{x^2 - a^2}$.

Illustration 1. $\int \sqrt{a^2 - x^2}\, dx$. Let $x = a \sin \theta$. Then $dx = a \cos \theta \, d\theta$, and

$$\begin{aligned}\int \sqrt{a^2 - x^2}\, dx = \int a^2 \cos^2 \theta \, d\theta &= \tfrac{1}{2}a^2(\theta + \tfrac{1}{2} \sin 2\theta) + C \\ &= \tfrac{1}{2}a^2(\theta + \sin \theta \cos \theta) + C \\ &= \tfrac{1}{2}a^2\left[\sin^{-1}\frac{x}{a} + \frac{x}{a^2}\sqrt{a^2 - x^2}\right] + C \\ &= \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a} + \tfrac{1}{2}x\sqrt{a^2 - x^2} + C.\end{aligned}$$

Illustration 2. $\int \sqrt{a^2 + x^2}\, x^3\, dx$. Let $x = a \tan \theta$. Then

$$\begin{aligned}\int \sqrt{a^2 + x^2}\, x^3\, dx &= a^5 \int \tan^3 \theta \sec^3 \theta \, d\theta \\ &= a^5 \int \tan^2 \theta \sec^2 \theta \tan \theta \sec \theta \, d\theta \\ &= a^5 \int (\sec^4 \theta - \sec^2 \theta) \tan \theta \sec \theta \, d\theta \\ &= a^5(\tfrac{1}{5} \sec^5 \theta - \tfrac{1}{3} \sec^3 \theta) + C \\ &= a^5\left[\frac{1}{5}\,\frac{(a^2 + x^2)^{\frac{5}{2}}}{a^5} - \frac{1}{3}\,\frac{(a^2 + x^2)^{\frac{3}{2}}}{a^3}\right] + C \\ &= \frac{(a^2 + x^2)^{\frac{5}{2}}}{5} - \frac{a^2(a^2 + x^2)^{\frac{3}{2}}}{3} + C.\end{aligned}$$

Illustration 3. $\int \frac{\sqrt{x^2 - a^2}}{x}\, dx$. Let $x = a \sec \theta$. Then $dx = a \sec \theta \tan \theta \, d\theta$, and

$$\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = a\int \frac{\sec\theta \tan^2\theta\, d\theta}{\sec\theta}$$

$$= a\int \tan^2\theta\, d\theta$$

$$= a(\tan\theta - \theta) + C$$

$$= a\left[\frac{\sqrt{x^2 - a^2}}{a} - \sec^{-1}\frac{x}{a}\right] + C$$

$$= \sqrt{x^2 - a^2} - a\sec^{-1}\frac{x}{a} + C.$$

It is to be noted that the result in all these cases is to be expressed in terms of the original variable of integration. For this purpose it may be found convenient to draw a right triangle with the sides properly lettered. Thus in *Illustration* 1 the substitution, $x = a \sin\theta$, was made. From Fig. 68, we obtain

$$\cos\theta = \frac{\sqrt{a^2 - x^2}}{a}.$$

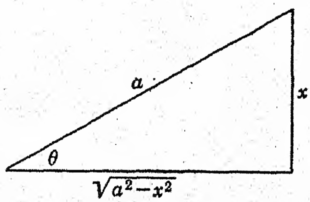

FIG. 68.

The substitutions used in these illustrations are summarized in the following table, which is not to be memorized. As soon as the method is understood the proper substitutions and subsequent transformations will suggest themselves.

Radical	Substitution	Radical becomes
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$a\tan\theta$

Expressions involving $\sqrt{ax^2 + bx + c}$ can frequently be integrated by completing the square under the radical sign and making a trigonometric substitution.

Illustration 4. $\displaystyle\int \frac{x\,dx}{\sqrt{3 + 2x - x^2}} = \int \frac{x\,dx}{\sqrt{4 - (x-1)^2}}.$

Let $x - 1 = 2 \sin \theta$. Then $x = 1 + 2 \sin \theta$ and $dx = 2 \cos \theta\, d\theta$. Hence

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{3 + 2x - x^2}} &= 2\int \frac{(1 + 2 \sin \theta) \cos \theta\, d\theta}{2 \cos \theta} \\ &= \int (1 + 2 \sin \theta)d\theta \\ &= \theta - 2 \cos \theta + C \\ &= \sin^{-1} \frac{x-1}{2} - \sqrt{3 + 2x - x^2} + C.\end{aligned}$$

To obtain $\cos \theta$ use the triangle of Fig. 69.

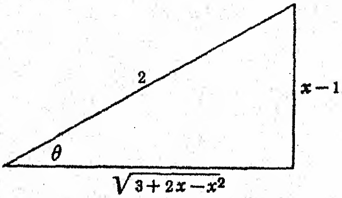

FIG. 69.

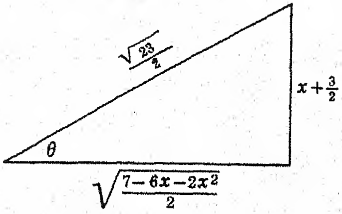

FIG. 70.

Illustration 5:

$$\int \frac{(2x-5)dx}{\sqrt{7 - 6x - 2x^2}} = \frac{1}{\sqrt{2}}\int \frac{(2x-5)dx}{\sqrt{\frac{23}{4} - (x + \frac{3}{2})^2}}.$$

Let

$$x + \tfrac{3}{2} = \frac{\sqrt{23}}{2} \sin \theta.$$

$$\begin{aligned}\int \frac{(2x-5)dx}{\sqrt{7 - 6x - 2x^2}} &= \frac{1}{\sqrt{2}}\int (\sqrt{23} \sin \theta - 8)d\theta \\ &= \frac{-1}{\sqrt{2}}(\sqrt{23} \cos \theta + 8\theta) + C \\ &= -\sqrt{7 - 6x - 2x^2} - 4\sqrt{2} \sin^{-1} \frac{2x+3}{\sqrt{23}} \\ &\qquad + C.\end{aligned}$$

To obtain $\cos\theta$ use the triangle of Fig. 70.

Exercises

1. $\int \sqrt{9 - x^2}\, dx.$

2. $\int \frac{dx}{x\sqrt{x^2 + 4}}.$

3. $\int \frac{dx}{(x^2 - 9)^{\frac{3}{2}}}.$

4. $\int \sqrt{16 - 9x^2}\, dx.$

5. $\int \frac{dx}{x^2\sqrt{x^2 - 4}}.$

6. $\int \frac{x^2\, dx}{\sqrt{9 - x^2}}.$

7. $\int \frac{dx}{x^2\sqrt{a^2 + x^2}}.$

8. $\int \frac{dx}{(9 - x^2)^{\frac{3}{2}}}.$

9. $\int \frac{dx}{x^3\sqrt{x^2 - 16}}.$

10. $\int \frac{dx}{x^2\sqrt{4 - x^2}}.$

11. $\int \frac{dx}{x^2\sqrt{5 - 2x^2}}.$

12. $\int \frac{dx}{x\sqrt{16 + 9x^2}}.$

13. $\int \frac{dx}{x(x^2 - 9)^{\frac{3}{2}}}.$

14. $\int \sqrt{7 - 5x^2}\, dx.$

15. $\int \frac{dx}{x^2\sqrt{3x^2 - 2}}.$

16. $\int \frac{dx}{(7 - x^2)^{\frac{3}{2}}}.$

17. $\int \frac{\sqrt{a^2 + x^2}}{x^2}\, dx.$

18. $\int \frac{dx}{(6x - 5 - x^2)^{\frac{3}{2}}}.$

19. $\int \frac{(2x + 5)dx}{(12 + 4x - x^2)^{\frac{3}{2}}}.$

20. $\int \frac{(3x + 2)dx}{(x^2 + 6x + 25)^{\frac{3}{2}}}.$

21. $\int \frac{(4x - 5)dx}{\sqrt{7 + 4x - 3x^2}}.$

105. Change of Limits of Integration. In working the preceding exercises by substitution of a new variable it was necessary to express the result of integration in terms of the original variable. In the case of definite integrals this last transformation can be avoided by changing the limits of integration.

Illustration 1. $\int_0^a x^2\sqrt{a^2 - x^2}\, dx.$ Let $x = a\sin\theta$. Then $dx = a\cos\theta\, d\theta$.

$$\text{When } x = 0, \sin\theta = 0 \text{ and } \theta = 0.$$
$$\text{When } x = a, \sin\theta = 1 \text{ and } \theta = \frac{\pi}{2}.$$

As x varies continuously from 0 to a, θ varies continuously from 0 to $\frac{\pi}{2}$. Hence we have

$$\begin{aligned}\int_0^a x^2\sqrt{a^2 - x^2}\,dx &= a^4\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta\\ &= a^4\left(\tfrac{1}{8}\theta - \tfrac{1}{32}\sin 4\theta\right)\Big|_0^{\frac{\pi}{2}}\\ &= \frac{\pi a^4}{16}.\end{aligned}$$

Illustration 2. $\displaystyle\int_0^a \frac{x^3\,dx}{\sqrt{a^2 + x^2}}$. Let $x = a\tan\theta$. Then $dx = a\sec^2\theta\,d\theta$.

$$\text{When } x = 0, \tan\theta = 0 \text{ and } \theta = 0.$$
$$\text{When } x = a, \tan\theta = 1 \text{ and } \theta = \frac{\pi}{4}.$$

As x varies continuously from 0 to a, θ varies continuously from 0 to $\frac{\pi}{4}$. Hence we have

$$\begin{aligned}\int_0^a \frac{x^3\,dx}{\sqrt{a^2 + x^2}} &= a^3\int_0^{\frac{\pi}{4}} \tan^3\theta\sec\theta\,d\theta = a^3\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right)\Big|_0^{\frac{\pi}{4}}\\ &= \tfrac{1}{3}a^3(2 - \sqrt{2}).\end{aligned}$$

Illustration 3. $\displaystyle\int_0^a \sqrt{a^2 - x^2}\,dx$. By using the substitution $x = a\sin\theta$ we obtain

$$\int_0^a \sqrt{a^2 - x^2}\, dx = a^2 \int_0^{\frac{\pi}{2}} \cos^2 \theta\, d\theta$$

$$= \tfrac{1}{2}a^2(\theta + \tfrac{1}{2}\sin 2\theta)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi a^2}{4}.$$

The above integral is of frequent occurrence in the applications of the calculus. The integrand, $\sqrt{a^2 - x^2}$, is represented graphically by the ordinates of a circle of radius a, center at the origin. The integral then represents the area of one-quarter of this circle (see §§**65** and **66**). The value of any integral of this form may be written down at once. Thus,

$$\int_0^7 \sqrt{49 - x^2}\, dx = \frac{49\pi}{4}$$

$$\int_0^{\sqrt{3+z^2}} \sqrt{3 + z^2 - x^2}\, dx = \frac{\pi(3 + z^2)}{4},$$

z being treated as a constant.

Exercises

1. $\int_0^5 x^3\sqrt{25 - x^2}\, dx.$

2. $\int_0^1 (2 - x^2)^{\frac{3}{2}} dx.$

3. $\int_0^4 \sqrt{16 - x^2}\, dx.$

4. $\int_0^3 \frac{dx}{(x^2 + 9)^2}.$

5. $\int_3^{3\sqrt{2}} \frac{x^3\, dx}{\sqrt{x^2 - 9}}.$

6. $\int_0^3 \sqrt{9 - x^2}\, dx.$

7. $\int_2^4 \frac{dx}{x^2\sqrt{x^2 - 4}}.$

8. $\int_0^5 \sqrt{25 - x^2}\, dx.$

9. $\int_3^5 \sqrt{6x - x^2 - 5}\, dx.$

10. $\int_0^5 \frac{dx}{(x^2 + 25)^{\frac{3}{2}}}.$

11. $\int_2^5 \frac{(2x + 3)dx}{\sqrt{5 + 4x - x^2}}.$

12. $\int_0^6 \sqrt{36 - x^2}\, dx.$

13. $\displaystyle\int_2^7 \frac{(3x + 2)dx}{\sqrt{2x^2 - 8x + 58}}.$

14. $\displaystyle\int_0^2 \frac{x^3dx}{\sqrt{x^2 + 4}}.$

15. $\displaystyle\int_0^8 \sqrt{64 - x^2}\, dx.$

16. $\displaystyle\int_0^2 \sqrt{16 - x^2}\, dx.$

17. $\displaystyle\int_{2\sqrt{2}}^4 \frac{dx}{x(x^2 - 4)^{\frac{3}{2}}}.$

18. $\displaystyle\int_3^6 \frac{dx}{x^2(x^2 - 9)^{\frac{1}{2}}}.$

106. Integration by Parts. The differential of the product of two functions u and v is

$$d(uv) = u\,dv + v\,du. \tag{1}$$

Integrating, we obtain

$$uv = \int u\,dv + \int v\,du,$$

from which

$$\int \mathbf{u\,dv} = \mathbf{uv} - \int \mathbf{v\,du}. \tag{2}$$

This equation is known as the formula for *integration by parts.* It makes the integration of $u\,dv$ depend upon that of the integration of dv and of $v\,du$.

It can frequently be applied when other methods fail.

Illustration 1. $\displaystyle\int x \log x\, dx.$ Let $\log x = u$ and $x\,dx = dv$. The application of (2) gives

$$\int x \log x\, dx = \tfrac{1}{2}x^2 \log x - \tfrac{1}{2}\int x^2 \frac{1}{x}dx$$
$$= \tfrac{1}{2}x^2 \log x - \tfrac{1}{4}x^2 + C.$$

Illustration 2. $\displaystyle\int xe^{3x}\, dx.$ Let $e^{3x}\, dx = dv$ and $x = u$. The application of (2) gives

$$\int xe^{3x}\,dx = \tfrac{1}{3}xe^{3x} - \tfrac{1}{3}\int e^{3x}\,dx$$
$$= \tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x} + C$$
$$= \tfrac{1}{9}e^{3x}(3x - 1) + C.$$

If we had let $x\,dx = dv$ and $e^{3x} = u$, we should have obtained a more complicated expression to integrate than that with which we started.

Exercises

1. $\int x \sin 2x\,dx.$

2. $\int x^3 \log x\,dx.$

3. $\int x \cos 3x\,dx.$

4. $\int \sin^{-1} x\,dx.$

5. $\int x^2 \sin x\,dx.$ (Apply formula (2) twice in succession.)

6. $\int \tan^{-1} x\,dx.$

7. $\int x^2 e^{3x}\,dx.$

8. $\int x \tan^{-1} x\,dx.$

9. $\int x^2 \cos x\,dx.$

10. $\int x^n \log x\,dx.$

11. $\int \cos x \log (\sin x)dx.$

12. $\int \dfrac{x}{\sqrt{2 + x}}\,dx.$

13. $\int x\sqrt{x - 2}\,dx.$

14. $\int x \sin^{-1} x\,dx.$

15. $\int_0^{\frac{\pi}{2}} x \cos x\,dx.$

16. $\int_0^1 x \tan^{-1} x\,dx.$

17. $\int_0^1 xe^x\,dx.$

18. $\int x \cos^3 x\,dx.$

107. The Integrals $\int e^{ax} \sin nx\,dx$, $\int e^{ax} \cos nx\,dx$. Let $u = \sin nx$ and $dv = e^{ax}\,dx$. Then

$$\int e^{ax} \sin nx\, dx = \frac{1}{a} e^{ax} \sin nx - \frac{n}{a} \int e^{ax} \cos nx\, dx.$$

A second integration by parts with $u = \cos nx$ and $dv = e^{ax}\, dx$ gives

$$\int e^{ax} \sin nx\, dx = \frac{1}{a} e^{ax} \sin nx - \frac{n}{a^2} e^{ax} \cos nx - \frac{n^2}{a^2} \int e^{ax} \sin nx\, dx.$$

The last term is equal to the integral in the first member multiplied by $\frac{n^2}{a^2}$. On transposing this term to the first member we obtain

$$\frac{a^2 + n^2}{a^2} \int e^{ax} \sin nx\, dx = \frac{e^{ax}}{a^2} (a \sin nx - n \cos nx) + C.$$

Then

$$\int \mathbf{e^{ax} \sin nx\, dx} = \frac{\mathbf{e^{ax}}}{\mathbf{a^2 + n^2}} \mathbf{(a \sin nx - n \cos nx) + C'}. \qquad (1)$$

The student will show in a similar way that

$$\int \mathbf{e^{ax} \cos nx\, dx} = \frac{\mathbf{e^{ax}}}{\mathbf{a^2 + n^2}} \mathbf{(n \sin nx + a \cos nx) + C}. \qquad (2)$$

Equation (1) can be written in the form

$$\int \mathbf{e^{ax} \sin nx\, dx} = \frac{\mathbf{e^{ax}}}{\sqrt{\mathbf{a^2 + n^2}}} \mathbf{\sin (nx - \alpha) + C'}, \qquad (3)$$

where

$$\cos \alpha = \frac{a}{\sqrt{a^2 + n^2}}$$

and

$$\sin \alpha = \frac{n}{\sqrt{a^2 + n^2}}.$$

Equation (2) can be written in the form

$$\int e^{ax} \cos nx \, dx = \frac{e^{ax}}{\sqrt{a^2 + n^2}} \cos (nx - \alpha) + C, \qquad (4)$$

where $\cos \alpha$ and $\sin \alpha$ are given by the expressions written above.

Exercises

The student will work Exercises 1 to 5 by the method used in obtaining (1) and (2). In the remaining exercises he may obtain the results by substituting in (1) and (2) as formulas.

1. $\int e^{3t} \sin 4t \, dt.$
2. $\int e^{x} \cos x \, dx.$
3. $\int e^{x} \sin x \, dx.$
4. $\int e^{2x} \cos 3x \, dx.$
5. $\int e^{-2x} \sin 3x \, dx.$
6. $\int e^{-3t} \cos 5t \, dt.$
7. $\int e^{-0.2t} \cos 3t \, dt.$
8. $\int e^{-kt} \sin \omega t \, dt.$
9. $\int e^{-0.1t} \cos 4t \, dt.$
10. $\int e^{-0.3t} \sin 5t \, dt.$
11. Express a few of the results of Exercises 1 to 10 in the form (3) or (4) and find α.

108. $\int \sec^3 x \, dx$. This integral can be evaluated by a method similar to that used in the last article.

$$\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx.$$

Since $\tan^2 x = \sec^2 x - 1$,

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx.$$

Transposing the next to the last term to the first member, dividing by 2, and integrating the last term, we have

$$\int \sec^3 x\,dx = \tfrac{1}{2}\left[\sec x \tan x + \log(\sec x + \tan x)\right] + C.$$

Exercises

1. $\displaystyle\int \csc^3 x\,dx.$
2. $\displaystyle\int \sec^5 x\,dx.$
3. $\displaystyle\int \frac{x^2}{\sqrt{x^2 - a^2}}\,dx.$
4. $\displaystyle\int \frac{\sqrt{a^2 + x^2}}{x^3}\,dx.$
5. $\displaystyle\int \sqrt{x^2 - 16}\,dx.$
6. $\displaystyle\int \sqrt{a^2 + x^2}\,dx.$
7. $\displaystyle\int_0^2 \sqrt{x^2 + 4}\,dx.$
8. $\displaystyle\int_2^5 \sqrt{x^2 - 4x + 13}\,dx.$
9. $\displaystyle\int_2^4 \sqrt{x^2 - 4}\,dx.$
10. $\displaystyle\int_a^{2a} \frac{x^4}{\sqrt{x^2 - a^2}}\,dx.$

109. Wallis' Formulas. Formulas will now be derived which make it possible to write down at once the values of the following definite integrals to which many other definite integrals can be reduced by trigonometric substitution:

$$\int_0^{\frac{\pi}{2}} \sin^n \theta\,d\theta,$$

$$\int_0^{\frac{\pi}{2}} \cos^n \theta\,d\theta,$$

and

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \cos^m \theta\,d\theta,$$

where m and n are positive integers greater than 1.

It is suggested that the reader study the derivation of formula (1) and accept formula (2), at least at first, as derived in a similar way. Formulas (1) and (2) should be committed to memory.

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^{n-1} \theta \sin \theta \, d\theta.$$

Integration by parts gives

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta &= - \sin^{n-1} \theta \cos \theta \Big|_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \cos^2 \theta \, d\theta \\
&= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta (1 - \sin^2 \theta) \, d\theta \\
&= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \, d\theta - (n-1) \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta.
\end{aligned}$$

On transposing the last term and dividing by n we obtain

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} \theta \, d\theta.$$

This equation may be regarded as a reduction formula for expressing

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$$

in terms of an integral in which $\sin \theta$ occurs with its exponent diminished by 2. Applying this formula successively we obtain

$$\int_0^{\frac{\pi}{2}} \sin^n \theta\, d\theta = \frac{n-1}{n}\frac{n-3}{n-2}\int_0^{\frac{\pi}{2}} \sin^{n-4} \theta\, d\theta$$

$$= \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\int_0^{\frac{\pi}{2}} \sin^{n-6} \theta\, d\theta$$

$$= \begin{cases} \dfrac{(n-1)(n-3)\cdots 4\cdot 2}{n(n-2)\cdots 5\cdot 3}\displaystyle\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta & \text{if } n \text{ is odd.} \\ \dfrac{(n-1)(n-3)\cdots 3\cdot 1}{n(n-2)\cdots 4\cdot 2}\displaystyle\int_0^{\frac{\pi}{2}} d\theta & \text{if } n \text{ is even.} \end{cases}$$

$$\int_0^{\frac{\pi}{2}} \sin^n \theta\, d\theta = \begin{cases} \dfrac{(n-1)(n-3)\cdots 4\cdot 2}{n(n-2)\cdots 3\cdot 1} & \text{if } n \text{ is odd.} \\ \dfrac{(n-1)(n-3)\cdots 3\cdot 1}{n(n-2)\cdots 4\cdot 2}\dfrac{\pi}{2} & \text{if } n \text{ is even.} \end{cases}$$

From the fact that the integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

and

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx$$

represent the areas under the curves $y = \sin^n x$ and $y = \cos^n x$, respectively, between the limits $x = 0$ and $x = \frac{\pi}{2}$, it is clear from the graphs that

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \int_0^{\frac{\pi}{2}} \sin^n x\, dx.$$

The results obtained can be expressed in the single formula

$$\int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \frac{(n-1)(n-3) \cdots 2 \text{ or } 1}{n(n-2) \cdots 2 \text{ or } 1} \alpha, \quad (1)$$

where $\alpha = 1$ if n is odd, and $\alpha = \frac{\pi}{2}$ if n is even.

In a similar way we shall evaluate

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta.$$

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^{m-1} \theta \cos^n \theta \sin \theta \, d\theta$$

$$= -\frac{\sin^{m-1} \theta \cos^{n+1} \theta}{n+1}\Bigg|_0^{\frac{\pi}{2}} + \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2} \theta \cos^{n+2} \theta \, d\theta$$

$$= \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2} \theta \cos^n \theta (1 - \sin^2 \theta) \, d\theta$$

$$= \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2} \theta \cos^n \theta \, d\theta - \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta.$$

Transposing the last term to the left member of the equation,

$$\left[1 + \frac{m-1}{n+1}\right]\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \sin^{m-2} \theta \cos^n \theta \, d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{m-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^{m-2} \cos^n \theta \, d\theta.$$

Apply this formula successively and obtain

$$\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \, d\theta = \frac{m-1}{m+n}\,\frac{m-3}{m+n-2}\int_0^{\frac{\pi}{2}} \sin^{m-4}\theta \cos^n \theta \, d\theta$$

$$= \begin{cases} \dfrac{(m-1)(m-3)\cdots 1}{(m+n)(m+n-2)\cdots(n+2)}\displaystyle\int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta \text{ if } m \text{ is even.} \\[2ex] \dfrac{(m-1)(m-3)\cdots 2}{(m+n)(m+n-2)\cdots(n+3)}\displaystyle\int_0^{\frac{\pi}{2}} \sin\theta\cos^n\theta\, d\theta \text{ if } m \text{ is odd.} \end{cases}$$

$$= \begin{cases} \left.\begin{array}{ll} \dfrac{(m-1)(m-3)\cdots 1\cdot(n-1)(n-3)\cdots 1}{(m+n)(m+n-2)\cdots(n+2)(n)(n-2)\cdots 2}\,\dfrac{\pi}{2} & \text{if } n \text{ is even} \\[2ex] \dfrac{(m-1)(m-3)\cdots 1\cdot(n-1)(n-3)\cdots 2}{(m+n)(m+n-2)\cdots(n+2)(n)(n-2)\cdots 3} & \text{if } n \text{ is odd} \end{array}\right\} \text{ and } m \text{ is even.} \\[2ex] \dfrac{(m-1)(m-3)\cdots 2}{(m+n)(m+n-2)\cdots(n+3)(n+1)} \quad \text{if } n \text{ is either even or odd, and } m \text{ is odd.} \end{cases}$$

The right-hand member of the last formula of this group can be put in a form similar to the others by multiplying numerator and denominator by $(n-1)(n-3)\cdots 2$ or 1. It becomes

$$\frac{(m-1)(m-3)\cdots 2\cdot(n-1)(n-3)\cdots 2 \text{ or } 1}{(m+n)(m+n-2)\cdots(n+3)(n+1)(n-1)(n-3)\cdots 2 \text{ or } 1}.$$

These formulas for $\displaystyle\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\, d\theta$ can all be expressed in the single formula

$$\int_0^{\frac{\pi}{2}} \sin^m\theta \cos^n\theta\, d\theta = \frac{(m-1)(m-3)\cdots 2 \text{ or } 1\cdot(n-1)(n-3)\cdots 2 \text{ or } 1}{(m+n)\,(m+n-2)\cdots 2 \text{ or } 1}\,\alpha, \tag{2}$$

where $\alpha = 1$ unless m and n are both even, in which case $\alpha = \dfrac{\pi}{2}$.

Illustration 1. By formula (1),

$$\int_0^{\frac{\pi}{2}} \sin^9\theta\, d\theta = \frac{8\cdot 6\cdot 4\cdot 2}{9\cdot 7\cdot 5\cdot 3\cdot 1} = \frac{128}{315}.$$

Illustration 2.

$$\int_0^{\frac{\pi}{2}} \cos^4 \theta \, d\theta = \frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} = \frac{3\pi}{16}.$$

Illustration 3. By formula (2),

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^3 x \, dx = \frac{4\cdot 2\cdot 2}{8\cdot 6\cdot 4\cdot 2} = \frac{1}{24}.$$

Illustration 4.

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^4 x \, dx = \frac{4\cdot 2\cdot 3\cdot 1}{9\cdot 7\cdot 5\cdot 3\cdot 1} = \frac{8}{315}.$$

Illustration 5.

$$\int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x \, dx = \frac{5\cdot 3\cdot 1\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\frac{\pi}{2} = \frac{3\pi}{512}.$$

Exercises

1. $\int_0^{\frac{\pi}{2}} \sin^5 \theta \, d\theta.$

2. $\int_0^{\frac{\pi}{2}} \cos^{10} \theta \, d\theta.$

3. $\int_0^{\frac{\pi}{2}} \cos^9 \theta \, d\theta.$

4. $\int_0^{\frac{\pi}{2}} \sin^6 \theta \, d\theta.$

5. $\int_0^{\frac{\pi}{2}} \cos^6 \theta \, d\theta.$

6. $\int_0^{\frac{\pi}{2}} \sin^8 \theta \, d\theta.$

7. $\int_0^{\frac{\pi}{2}} \cos^{11} x \, dx.$

8. $\int_0^{\frac{\pi}{2}} \sin^7 \phi \, d\phi.$

9. $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x \, dx.$

10. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^5 x \, dx.$

11. $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x \, dx.$

12. $\int_0^{\frac{\pi}{2}} \sin^5 \phi \cos \phi \, d\phi.$

13. $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^8 x \, dx.$

14. $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^5 x \, dx.$

15. $\int_0^{a} (a^2 - x^2)^{\frac{3}{2}} \, dx.$

16. $\int_0^{a} x^2(a^2 - x^2)^{\frac{3}{2}} \, dx.$

17. $\int_0^{a} (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} \, dx.$

18. $\int_0^{a} x(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} \, dx.$

It is to be noted that

$$\int_0^{\pi} \sin^n x \, dx = 2\int_0^{\frac{\pi}{2}} \sin^n x \, dx, \text{ if } n \text{ is a positive integer}$$

and

$$\int_0^{\pi} \cos^n x \, dx = 0, \text{ if } n \text{ is odd}$$

$$= 2\int^{\frac{\pi}{2}} \cos^n x \, dx, \text{ if } n \text{ is even.}$$

These relations and others similar to them can be readily discovered by considering the graphs of the functions $y = \sin^n x$ and $y = \cos^n x$.

19. $\int_0^{\pi} \sin^4 \theta \, d\theta.$

20. $\int_0^{\pi} \cos^6 \theta \, d\theta.$

21. $\int_0^{\pi} \sin^5 \theta \, d\theta.$

22. $\int_0^{\pi} \cos^5 \theta \, d\theta.$

23. $\int_0^{2\pi} \sin^2 \theta \, d\theta.$

24. $\int_0^{2\pi} \sin^3 \theta \, d\theta.$

25. $\int_0^{\pi} a^2(1 - \cos \theta)^2 \, d\theta = 4a^2 \int_0^{\pi} \sin^4 \frac{\theta}{2} \, d\theta.$

Let $\theta' = \frac{\theta}{2}$. Then $d\theta = 2d\theta'$ and $\theta' = \frac{\pi}{2}$ when $\theta = \pi$, and $\theta' = 0$ when $\theta = 0$. Hence

$$a^2 \int_0^{\pi} (1 - \cos\theta)^2 \, d\theta = 8a^2 \int_0^{\frac{\pi}{2}} \sin^4 \theta' \, d\theta'.$$

Wallis' formula can now be applied.

This integral can also be easily evaluated without changing the variable. Thus

$$\int_0^{\pi} a^2(1 - \cos\theta)^2 \, d\theta = a^2 \int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta$$

$$= 2a^2 \int_0^{\frac{\pi}{2}} (1 + \cos^2\theta) d\theta$$

$$= 2a^2\left(\frac{\pi}{2} + \frac{1}{2}\frac{\pi}{2}\right) = \frac{3\pi a^2}{2}.$$

By transformations similar to the foregoing many integrals can be put into a form to which Wallis' formulas can be applied.

26. $\int_0^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^2 \theta' \, d\theta'.$

27. $\int_0^{\pi} (1 - \cos\theta)^3 \, d\theta.$

28. $\int_0^{\frac{\pi}{6}} \cos^2 3\theta \, d\theta.$

29. $\int_0^{\frac{\pi}{6}} \sin^4 3\theta \, d\theta.$

30. $\int_0^{\pi} (1 + \cos\theta)^2 \, d\theta.$

31. $\int_0^{\frac{\pi}{4}} \sin^4 2\theta \, d\theta.$

32. $\int_0^{\pi} (1 + \cos\theta)^3 \, d\theta.$

33. $\int_0^{2a} (2ax - x^2)^{\frac{3}{2}} \, dx.$

(Substitute $x = 2a\sin^2\theta$).

34. $\int_0^{2a} x\sqrt{2ax - x^2} \, dx.$

110. Integration of $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx$. Integrals of this form can be reduced by the substitution, $z = \tan \frac{x}{2}$. In making this substitution it is necessary to express $\sin x$, $\cos x$, and dx in terms of z. This is easily done as follows. (The student is advised to observe the method carefully, but not to memorize the results, as he can readily obtain them whenever needed.) Since

$$z = \tan \frac{x}{2},$$
$$x = 2 \tan^{-1} z,$$

and

$$dx = 2\frac{dz}{1 + z^2}.$$

Further,

$$\cos \frac{x}{2} = \frac{1}{\sec \frac{x}{2}} = \frac{1}{\sqrt{1 + \tan^2 \frac{x}{2}}} = \frac{1}{\sqrt{1 + z^2}}$$

and

$$\sin \frac{x}{2} = \tan \frac{x}{2} \cos \frac{x}{2} = \frac{z}{\sqrt{1 + z^2}}.$$

Then

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2z}{1 + z^2}$$

and

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1 - z^2}{1 + z^2}.$$

Illustration 1. $\int \frac{dx}{1 + 4\cos x}$. On making the substitution $z = \tan\frac{x}{2}$ we obtain, by using the values just found for $\cos x$ and dx in terms of z,[1]

$$\int \frac{\frac{2dz}{1+z^2}}{1 + 4\frac{1-z^2}{1+z^2}} = 2\int \frac{dz}{1 + z^2 + 4(1 - z^2)}$$

$$= 2\int \frac{dz}{5 - 3z^2}$$

$$= -\frac{2}{\sqrt{3}}\int \frac{\sqrt{3}\,dz}{3z^2 - 5}$$

$$= -\frac{2}{2\sqrt{3}\sqrt{5}}\log\frac{\sqrt{3}z - \sqrt{5}}{\sqrt{3}z + \sqrt{5}} + C$$

$$= \tfrac{1}{15}\sqrt{15}\log\frac{\sqrt{3}\tan\frac{x}{2} + \sqrt{5}}{\sqrt{3}\tan\frac{x}{2} - \sqrt{5}} + C.$$

Illustration 2. $\int \frac{dx}{5 - 3\sin x}$. Let $z = \tan\frac{x}{2}$. Then

$$\int \frac{dx}{5 - 3\sin x} = \int \frac{\frac{2dz}{1+z^2}}{5 - \frac{6z}{1+z^2}} = 2\int \frac{dz}{5 + 5z^2 - 6z}$$

$$= \tfrac{2}{5}\int \frac{dz}{z^2 - \frac{6}{5}z + 1} = \tfrac{2}{5}\int \frac{dz}{(z - \frac{3}{5})^2 + \frac{16}{25}}$$

$$= \tfrac{2}{5}\cdot\tfrac{5}{4}\tan^{-1}\frac{z - \frac{3}{5}}{\frac{4}{5}} + C = \tfrac{1}{2}\tan^{-1}\frac{5z - 3}{4} + C$$

$$= \tfrac{1}{2}\tan^{-1}\frac{5\tan\frac{x}{2} - 3}{4} + C.$$

[1] The student will derive these values in each problem worked in order to familiarize himself with the method.

Exercises

The student will find cos x, sin x, and dx in terms of the new variable in each of the exercises.

1. $\displaystyle\int \frac{dx}{3 + 5\cos x}$.

2. $\displaystyle\int \frac{dx}{5 - 3\cos x}$.

3. $\displaystyle\int \frac{dx}{4 - 5\sin x}$.

4. $\displaystyle\int \frac{dx}{5 - 3\sin 2x}$.

5. $\displaystyle\int \frac{dx}{4 - 5\cos 2x}$.

6. $\displaystyle\int \frac{dx}{3 - 5\cos x}$.

7. $\displaystyle\int \frac{\sin x\, dx}{\sin x + 2} = \int \left[1 - \frac{2}{\sin x + 2}\right] dx.$

8. $\displaystyle\int \frac{\cos x\, dx}{3 + 2\cos x}$.

111. Partial Fractions. A rational fraction is the quotient of two polynomials, *e.g.*,

$$\frac{a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n} = \frac{\phi(x)}{f(x)}. \tag{1}$$

If the degree of the numerator, m, is greater than or equal to the degree of the denominator, n, the fraction can be transformed by division into the sum of a polynomial and a fraction whose numerator is of lower degree than the denominator. In this case the division is always to be performed before applying the methods of this section.

The integration of a rational fraction cannot, in general, be accomplished by the methods which have been given if the degree of the denominator is greater than 2. Illustrations will now be given of a process by which a rational fraction can be expressed as the sum of fractions whose denominators are of either the first or second degrees.

Illustration 1.

$$\int \frac{x^2 + 2}{x^3 - 2x^2 - 9x + 18}\, dx.$$

Factoring the denominator,

$$x^3 - 2x^2 - 9x + 18 = (x - 2)(x - 3)(x + 3).$$

Assume

$$\frac{x^2 + 2}{x^3 - 2x^2 - 9x + 18} = \frac{A}{x - 2} + \frac{B}{x - 3} + \frac{C}{x + 3},$$

where A, B, and C are to be so determined that this equation shall be satisfied for all values of x. Clearing of fractions,

$$\begin{aligned} x^2 + 2 &= Ax^2 - 9A + Bx^2 + Bx - 6B + Cx^2 - 5Cx + 6C \\ &= (A + B + C)x^2 + (B - 5C)x - 9A - 6B + 6C. \end{aligned}$$

On equating the coefficients[1] of x^2, x, x^0, we obtain the following three equations for the determination of A, B, and C:

$$\begin{aligned} A + B + C &= 1. \\ B - 5C &= 0. \\ -9A - 6B + 6C &= 2. \end{aligned}$$

From these equations

$$\begin{aligned} A &= -\tfrac{6}{5}. \\ B &= \tfrac{11}{6}. \\ C &= \tfrac{11}{30}. \end{aligned}$$

Hence

$$\frac{x^2 + 2}{x^3 - 2x^2 - 9x + 18} = \frac{-6}{5(x - 2)} + \frac{11}{6(x - 3)} + \frac{11}{30(x + 3)}$$

[1] In applying this process use is made of the fact that if two polynominals in x are identically equal, the coefficients of like powers of x are equal. Thus, given the identity

$$\alpha_0 x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1} x + \alpha_n = \beta_0 x^n + \beta_1 x^{n-1} + \cdots + \beta_{n-1} x + \beta_n,$$

then

$$\begin{aligned} \alpha_0 &= \beta_0 \\ \alpha_1 &= \beta_1 \\ \cdots &= \cdots \\ \alpha_n &= \beta_n. \end{aligned}$$

and

$$\int \frac{x^2 + 2}{x^3 - 2x^2 - 9x + 18}dx = -\tfrac{6}{5}\int \frac{dx}{x - 2} + \tfrac{11}{6}\int \frac{dx}{x - 3} + \tfrac{11}{30}\int \frac{dx}{x + 3}$$

$$= -\tfrac{6}{5} \log (x - 2) + \tfrac{11}{6} \log (x - 3) + \tfrac{11}{30} \log (x + 3) + C.$$

Short Method. The foregoing method of determining the values of A, B, . . . by equating coefficients of like powers of x, is perfectly general. However, a shorter method can sometimes be used. Thus in the illustration just given write the result of clearing of fractions in the form

$$x^2 + 2 = A(x - 3)(x + 3) + B(x - 2)(x + 3) + C(x - 2)(x - 3).$$

Since this relation is true for all values of x, it is true for $x = 2$. On setting $x = 2$, we obtain

$$6 = -5A.$$

Hence

$$A = -\tfrac{6}{5}.$$

On setting $x = 3$, we obtain

$$11 = 6B.$$

Hence

$$B = \tfrac{11}{6}.$$

On setting $x = -3$, we obtain

$$11 = 30C.$$

Hence

$$C = \tfrac{11}{30}.$$

Illustration 2.

$$\int \frac{x^2 + 1}{(x + 1)(x - 1)^3}dx.$$

Let

$$\frac{x^2+1}{(x+1)(x-1)^3} = \frac{A}{x+1} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^2} + \frac{D}{x-1}.$$

On clearing of fractions,

$$x^2 + 1 = A(x-1)^3 + B(x+1) + C(x-1)(x+1) + D(x-1)^2(x+1),$$

or

$$x^2 + 1 = Ax^3 - 3Ax^2 + 3Ax - A + Bx + B + Cx^2 - C + Dx^3 - Dx^2 - Dx + D.$$

In the first form put $x = 1$. Then

$$B = 1.$$

In the first form put $x = -1$. Then

$$-8A = 2.$$

Hence

$$A = -\tfrac{1}{4}.$$

Equating coefficients of x^3 in the second form,

$$A + D = 0.$$

Hence

$$D = -A = \tfrac{1}{4}.$$

Equating coefficients of x^2 in the second form,

$$-3A + C - D = 1.$$

Hence

$$C = 1 - \tfrac{3}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

Consequently,

$$\frac{x^2+1}{(x+1)(x-1)^3} = \frac{-1}{4(x+1)} + \frac{1}{(x-1)^3} + \frac{1}{2(x-1)^2} + \frac{1}{4(x-1)}$$

and

$$\int \frac{x^2+1}{(x+1)(x-1)^3}\,dx = -\tfrac{1}{4}\int \frac{dx}{x+1} + \int \frac{dx}{(x-1)^3}$$
$$+ \tfrac{1}{2}\int \frac{dx}{(x-1)^2} + \tfrac{1}{4}\int \frac{dx}{x-1}$$
$$= -\tfrac{1}{4}\log(x+1) - \frac{1}{2(x-1)^2} - \frac{1}{2(x-1)} + \tfrac{1}{4}\log(x-1) + C$$
$$= \log\sqrt[4]{\frac{x-1}{x+1}} - \frac{x}{2(x-1)^2} + C.$$

Illustration 3.

$$\int \frac{3x^2-2x+2}{(x-1)(x^2-4x+13)}dx.$$

Let

$$\frac{3x^2-2x+2}{(x-1)(x^2-4x+13)} = \frac{A}{x-1} + \frac{Bx+C}{x^2-4x+13}.$$

Clearing of fractions,

$$3x^2-2x+2 = A(x^2-4x+13) + Bx(x-1) + C(x-1),$$

or

$$3x^2-2x+2 = Ax^2 - 4Ax + 13A + Bx^2 - Bx + Cx - C.$$

In the first form put $x = 1$. We obtain

$$3 = 10A.$$

Hence

$$A = \tfrac{3}{10}.$$

Equating the constant terms in the second form,

$$13A - C = 2.$$

Hence

$$\tfrac{39}{10} - C = 2$$

and

$$C = \tfrac{19}{10}.$$

Equating the coefficients of x^2 in the second form,

$$A + B = 3.$$

Hence

$$B = 3 - \tfrac{3}{10} = \tfrac{27}{10}.$$

Consequently,

$$\int \frac{3x^2 - 2x + 2}{(x-1)(x^2 - 4x + 13)}dx = \tfrac{3}{10}\int \frac{dx}{x-1} + \tfrac{1}{10}\int \frac{27x + 19}{x^2 - 4x + 13}dx$$

$$= \tfrac{3}{10} \log (x - 1) + \tfrac{27}{20}\int \frac{(2x - 4)\, dx}{x^2 - 4x + 13} + \tfrac{73}{10}\int \frac{dx}{(x-2)^2 + 9}$$

$$= \tfrac{3}{10} \log (x - 1) + \tfrac{27}{20} \log (x^2 - 4x + 13) + \tfrac{73}{30} \tan^{-1} \frac{x-2}{3} + C.$$

Illustration 4.

$$\int \frac{2x\, dx}{(1 + x)(1 + x^2)^2}.$$

Let

$$\frac{2x}{(1 + x)(1 + x^2)^2} = \frac{A}{1 + x} + \frac{Bx + C}{(1 + x^2)^2} + \frac{Dx + E}{(1 + x^2)}.$$

In *Illustrations* 1 to 4 a fraction was broken up into "partial fractions." The denominators were the factors of the denominator of the given fraction. In *Illustrations* 1 and 2 factors were all real linear factors, while in *Illustrations* 3 and 4 there were also factors of the second degree which could not be factored into two real linear factors. The method of procedure will be further indicated by the following examples. They will be grouped under the numbers I, II, III, and IV, corresponding to *Illustrations* 1, 2, 3, and 4.

I. Factors of denominator linear, none repeated.

$$(a)\quad \frac{x^2+5}{(x-1)(x+1)(x-3)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x-3}.$$

$$(b)\quad \frac{x^2+2x+7}{(x+4)(2x+3)(x-2)(3x+1)}=\frac{A}{x+4}+\frac{B}{2x+3}+\frac{C}{x-2}+\frac{D}{3x+1}.$$

II. Factors of denominator linear, some repeated.

$$(a)\quad \frac{x^2+2x+5}{(x-2)^2(x-3)^3(x+1)}=\frac{A}{(x-2)^2}+\frac{B}{x-2}+\frac{C}{(x-3)^3}+\frac{D}{(x-3)^2}+\frac{E}{x-3}+\frac{F}{x+1}.$$

$$(b)\quad \frac{x^2+4x-2}{(2x+1)^3(x+3)(x-4)^2}=\frac{A}{(2x+1)^3}+\frac{B}{(2x+1)^2}+\frac{C}{2x+1}+\frac{D}{x+3}+\frac{E}{(x-4)^2}+\frac{F}{x-4}.$$

III. Denominator contains factors of second degree, none repeated.

$$(a)\quad \frac{x^2+7x+3}{(x^2+4)(x-2)}=\frac{Ax+B}{x^2+4}+\frac{C}{x-2}.$$

$$(b)\quad \frac{x^3-3x+5}{(x^2+2)(x^2-4x+7)(x+3)}=\frac{Ax+B}{x^2+2}+\frac{Cx+D}{x^2-4x+7}+\frac{E}{x+3}.$$

$$(c)\quad \frac{x^2+2x-5}{(x^2+7)(x-2)^2}=\frac{Ax+B}{x^2+7}+\frac{C}{(x-2)^2}+\frac{D}{x-2}.$$

IV. Denominator contains factors of second degree, some repeated.

$$(a)\quad \frac{x^3+2x^2+5}{(x^2+2x+10)^2(x^2+3)(x+2)}=\frac{Ax+B}{(x^2+2x+10)^2}+\frac{Cx+D}{x^2+2x+10}+\frac{Ex+F}{x^2+3}+\frac{G}{x+2}.$$

Exercises

1. $\displaystyle\int \frac{4 - x}{x^2 + x - 2}dx.$

2. $\displaystyle\int \frac{2x^2 - 9x + 6}{x^3 - 5x^2 + 6x}dx.$

3. $\displaystyle\int \frac{2x^2 - 9x + 6}{x^2 - x^3}dx.$

4. $\displaystyle\int \frac{45 - 19x}{(x + 4)(2x - 3)}dx.$

5. $\displaystyle\int \frac{(5x - 12)dx}{(x + 2)(x^2 - x + 5)}.$

6. $\displaystyle\int \frac{(3x^2 - 6x - 3)dx}{x^3 - 2x^2 - 5x + 6}.$

7. $\displaystyle\int \frac{(x^4 + 2x^3 + 5x^2 + 5x + 2)dx}{x(x^2 + 1)^2}.$

8. $\displaystyle\int \frac{(4x^2 + 10x - 16)dx}{x^3 - 4x}.$

CHAPTER XII

APPLICATIONS OF THE PROCESS OF INTEGRATION. IMPROPER INTEGRALS

112. In this section a brief summary and review of the applications of the process of integration will be given. Some of the exercises lead to integrals that could not have been evaluated earlier in the course. The following definite integrals representing area, volume, etc., are not to be memorized. In each exercise a figure is to be drawn and the element of integration is to be determined from the figure.

1. *Area under a Plane Curve; Rectangular Coordinates.*

$$A = \int_a^b f(x)dx.$$

(See §**65**, and Fig. 39.)

2. *Area; Polar Coordinates.*

$$A = \tfrac{1}{2}\int_\alpha^\beta \rho^2\, d\theta.$$

(See §**99**, and Fig. 67.)

3. *Length of Arc of a Plane Curve; Rectangular Coordinates.*

$$\begin{aligned} s &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \end{aligned}$$

(See §**70**, and Fig. 43.)

4. *Length of Arc; Polar Coordinates.*

$$s = \int_{\alpha}^{\beta} \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2} d\theta$$
$$= \int_{\rho_1}^{\rho_2} \sqrt{1 + \rho^2\left(\frac{d\theta}{d\rho}\right)^2} d\rho.$$

(See §**98**, and Fig. 65.)

5. *Volume of a Solid of Revolution.*

$$V = \int_a^b \pi y^2 \, dx.$$

(See §**68**, and Fig. 40.)

6. *Surface of a Solid of Revolution.*

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
$$= 2\pi \int_{x=a}^{x=b} y \, ds$$
$$= 2\pi \int_{y_1}^{y_2} y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$
$$= 2\pi \int_{y=y_1}^{y=y_2} y \, ds.$$

(See §**71**, and Fig. 40.)

7. *Water Pressure on a Vertical Surface* (see §**73**).

8. *Work Done by a Variable Force* (see §**69**).

Exercises

1. Find the area bounded by the curve $y = 6x - 5 - x^2$ and the X-axis.

2. Find the volume of the ellipsoid of revolution generated by revolving the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ about the X-axis. About the Y-axis.

3. Find the length of one quadrant of the circle $x^2 + y^2 = 25$. Also use the parametric equations $x = 5 \cos \theta$, $y = 5 \sin \theta$.

4. Find the area bounded by one loop of $\rho = a \cos^2 \theta$.

5. Find the area of the surface of the sphere generated by revolving the circle $x^2 + y^2 = 100$ about the X-axis. Find the area of the same surface using the parametric equations $x = 10 \cos \theta$, $y = 10 \sin \theta$.

6. A trough 4 feet deep and 2 feet wide at the top has a parabolic cross section. Find the force due to water pressure on one end when the trough is full of water.

7. A force of 250 pounds will stretch a spring from its normal length of 20 inches to a length of 22 inches. Find the work done in stretching the spring from a length of 21 inches to a length of 23 inches.

8. Find the length of $\rho = 4 \cos \theta$.

9. Find the area of one quadrant of the ellipse $x = 5 \cos \theta$, $y = 3 \sin \theta$.

10. Find the area bounded by one loop of the lemniscate $\rho^2 = 9 \cos 2\theta$.

11. Find the area of the loop of $\rho = 2 - \sec \theta$.

12. Find the area bounded by the hypocycloid $x = 4 \cos^3 \theta$, $y = 4 \sin^3 \theta$.

13. Find the area of the loop of $\rho = a \cot \dfrac{\theta}{2}$.

14. Find the length of the cardioid $\rho = 5(1 + \cos \theta)$.

15. Find the area of the surface generated by revolving a quadrant of a circle, whose radius is a, about a tangent at one extremity.

16. A gas is expanding in accordance with Boyle's law, $pv = K$. Find the work done by the gas in expanding from a volume of 4 cubic feet at a pressure of 40 pounds per square inch to a volume of 6 cubic feet. Calculate the result correct to four significant figures.

17. In increasing the length of a bar by 2 inches, 200 inch-pounds of work is done. Find the work done in increasing its length by 4 inches.

18. Find the area of the surface generated by revolving about the X-axis the arc of the curve $y^2 = 2ax$ which lies between $x = 0$ and $x = a$.

19. Find the normal length of a spring if it is known that 600 inch-pounds and 1200 inch-pounds of work are done, respectively, in stretching the spring from a length of 32 inches to a length of 34 inches, and from a length of 33 inches to a length of 35 inches.

20. Find the normal length of a spring if 200 inch-pounds and 1250 inch-pounds of work are done in stretching it to a length of 52 inches and to a length of 55 inches, respectively.

21. Find the length of $\rho = e^{-2\theta}$ from $\theta = 0$ to $\theta = \alpha$. Find the limit of the result as α becomes infinite.

22. Find the work done by the gas referred to in Exercise 16, but expanding in accordance with the adiabatic law $pv^{1.4} = K$. Calculate the result correct to three significant figures.

23. Find the entire length of the curve whose parametric equations are $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

24. Find the volume of a frustum of a right circular cone of altitude h, the radii of the bases being r and R.

25. Find the lateral area of the frustum of a right circular cone, the altitude being h and the radii of the bases r and R. Express the result in terms of the slant height and the radii of the bases.

26. A cylindrical tank 8 feet in diameter is lying on the ground with its axis horizontal. It is half full of water. Find the force due to water pressure on one end of the tank.

27. Find the area outside $\rho = 5$ and inside $\rho = 2 \cos \theta + 4$.

28. Find the force due to water pressure on a trapezoidal gate closing a channel containing water 9 feet deep, the upper and lower bases of the wet surface being 20 and 14 feet, respectively.

29. Find the length of $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ between the points whose abscissas are 0 and 2.

30. Find the volume generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the X-axis. Also use the parametric equations $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

31. Find the area of the small loop of $\rho = 1 + 2 \cos \theta$.

32. Find the area of the surface generated by revolving the cardioid, $\rho = a(1 + \cos \theta)$, about the polar axis.

33. Find the area of the surface generated by revolving about the X-axis the portion of the arc of the catenary $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ which lies between the points whose abscissas are 0 and a.

34. Find the area bounded by the curve $x = 5 \cos^3 \theta$, $y = 4 \sin^3 \theta$.

35. Find the length of that portion of $y = x^2$ which lies between $x = 0$ and $x = 3$.

36. Find the area of the surface generated by revolving one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, about the X-axis. About the Y-axis. About the tangent at the highest point.

37. A solid is generated by a variable equilateral triangle moving with its plane perpendicular to the X-axis. A side of the triangle

in a plane at a distance x from the origin is $3x^2$. Find the volume of the solid lying between the sections $x = 1$ and $x = 3$.

38. Find the length of arc of the parabola $\rho = \frac{5}{2} \sec^2 \frac{\theta}{2}$ between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

39. Find the volume of the solid generated by revolving one arch of the cycloid, $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, about the X-axis. About the tangent at the highest point.

40. Find the length of one quadrant of the curve $x = a \cos^3 \phi$, $y = b \sin^3 \phi$.

41. Find the length of one loop of $\rho = a \cos^2 \theta$.

42. Find one of the smaller areas bounded by $y^2 = 2x$ and $y^2 = 6x - x^2$.

43. Find the work done in pumping the water out of a cistern 15 feet deep, in which the water stands at a depth of 9 feet, if the cistern is a paraboloid of revolution whose diameter at the surface of the ground is 10 feet and if the water is delivered at a height of 2 feet above the ground.

44. Find the area of one loop of $\rho = 5 \sin 6\theta$.

45. Find the surface generated by revolving one loop of $\rho^2 = a^2 \cos 2\theta$ about the polar axis.

46. The density of a right circular cone varies as the distance from the vertex. Find the mass of the cone if the altitude is h and the radius of the base is r.

47. The end of a water main 5 feet in diameter, leading from a reservoir, is closed by a vertical gate. Find the force due to water pressure on the gate if the surface of the water in the reservoir is 30 feet above the center of the gate.

48. Find the volume of the anchor ring generated by revolving the circle $x^2 + (y - 5)^2 = 4$ about the X-axis.

49. Set up the integral representing the length of one quadrant of the ellipse $x = a \cos \theta$, $y = b \sin \theta$.

50. Find the volume of one of the wedges cut from a right circular cylinder 12 inches in diameter by two planes, one of which is perpendicular to the axis of the cylinder, the other of which is inclined to the former at an angle of 45° and meets it in a diameter of the cylinder.

51. Find the work done in pumping the water from a hemispherical cistern 12 feet in diameter, if the water is 4 feet deep and is delivered at a height of 2 feet above the ground.

52. Find the area bounded by the parabola $\rho = \frac{5}{1 + \cos\theta} = \frac{5}{2}\sec^2\frac{\theta}{2}$ and the radii $\theta = 0$ and $\theta = \frac{\pi}{2}$.

53. Find the area bounded by one loop of $y^2 = x^2 - x^4$.

54. Find the force due to water pressure on a vertical semi-elliptical gate, the axes being 10 and 8 feet, respectively, if the major axis lies in the surface of the water. If the major axis lies 6 feet below the surface of the water.

113. Improper Integrals. Consider the integral

$$\int_1^7 \frac{1}{\sqrt{x-1}}\, dx.$$

In defining the definite integral, $\int_a^b f(x)\, dx$, §**65**, the assumption was made that $f(x)$ was a finite continuous function at the limits $x = a$ and $x = b$ as well as at all intermediate points, and the evaluation of this integral was based on the area under the curve $y = f(x)$.

In the case of the integral under consideration

$$f(x) = \frac{1}{\sqrt{x-1}}$$

becomes infinite at the lower limit. The area under the curve

$$y = \frac{1}{\sqrt{x-1}}$$

between the ordinates $x = 1$ and $x = 7$ has no meaning. In fact, the integral in question has no meaning in accordance with the definition of a definite integral already given. A new definition is necessary. We define

$$\int_1^7 \frac{1}{\sqrt{x-1}}\, dx$$

as

$$\lim_{\eta \to 0} \int_{1+\eta}^{7} \frac{1}{\sqrt{x-1}}\,dx,$$

where η is a small positive number, if this limit exists. Otherwise the integral has no meaning. Now,

$$\lim_{\eta \to 0} \int_{1+\eta}^{7} \frac{1}{\sqrt{x-1}}\,dx = \lim_{\eta \to 0} (2\sqrt{x-1})\Big|_{1+\eta}^{7}$$
$$= \lim_{\eta \to 0} (2\sqrt{6} - 2\sqrt{\eta}) = 2\sqrt{6}.$$

Since the limit exists we say that

$$\int_{1}^{7} \frac{1}{\sqrt{x-1}}\,dx = 2\sqrt{6}.$$

Graphically, this means the limit as η approaches zero of the area under the curve $y = \dfrac{1}{\sqrt{x-1}}$ between the ordinates $x = 1 + \eta$ and $x = 7$ exists and is equal to $2\sqrt{6}$.

Exercise 1. Show that

$$\int_{1}^{7} \frac{dx}{(x-1)^n}$$

exists if $0 < n < 1$.

On the other hand, when $n = 1$,

$$\lim_{\eta \to 0} \int_{1+\eta}^{7} \frac{1}{x-1}\,dx = \lim_{\eta \to 0} \log (x-1)\Big|_{1+\eta}^{7}$$
$$= \lim_{\eta \to 0} (\log 6 - \log \eta) = \lim_{\eta \to 0} \log \frac{6}{\eta}.$$

This limit does not exist and consequently we say that

$$\int_{1}^{7} \frac{1}{x-1}\,dx$$

has no meaning or does not exist.

Graphically, this means that the area under the curve

$$y = \frac{1}{x - 1}$$

between the ordinates $x = 1 + \eta$ and $x = 7$ increases without limit as η approaches zero.

Exercise 2. Show that

$$\int_1^7 \frac{dx}{(x - 1)^n}$$

does not exist if $n \geqq 1$. (Note that the case $n = 1$ has just been considered.) If $n < 0$, no question as to the meaning of the integral can arise. Why?

A definite integral in which the function to be integrated becomes infinite at the upper limit is treated in the same way. Thus

$$\int_0^1 \frac{dx}{\sqrt{1 - x}}$$

is defined as

$$\lim_{\eta \to 0} \int_0^{1-\eta} \frac{dx}{\sqrt{1 - x}}$$

where η is a positive number, if this limit exists.

Exercise 3. Show that

$$\int_0^1 \frac{dx}{(1 - x)^n}$$

has a meaning *in accordance with this definition* if $0 < n < 1$, and that it has no meaning if $n \geqq 1$. If $n < 0$, no question can arise as to the meaning of the integral.

It is easy to see how to proceed in case the function under the integral sign becomes infinite at a point within the interval of integration. Thus

$$\int_0^7 \frac{dx}{(x - 1)^n},$$

where n is a positive number, will have a meaning if each of the integrals

$$\int_0^1 \frac{dx}{(x-1)^n} \quad \text{and} \quad \int_1^7 \frac{dx}{(x-1)^n}$$

has a meaning.

Exercises

Evaluate the following integrals if they have a meaning:

1. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}.$
2. $\displaystyle\int_0^1 \frac{dx}{x^2}.$
3. $\displaystyle\int_0^1 \frac{dx}{x}.$
4. $\displaystyle\int_{-1}^{+1} \frac{dx}{\sqrt{x+1}}.$
5. $\displaystyle\int_{-1}^{+1} \frac{dx}{x^2}.$
6. $\displaystyle\int_0^a \frac{dx}{\sqrt{a-x}}.$
7. $\displaystyle\int_0^a \frac{dx}{\sqrt{a^2-x^2}}.$
8. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$
9. $\displaystyle\int_{\frac{4}{3}}^5 \frac{dx}{3x-4}.$
10. $\displaystyle\int_{-1}^{+1} \frac{dx}{x^{\frac{2}{3}}}.$
11. $\displaystyle\int_0^5 \frac{dx}{(x-2)^{\frac{2}{3}}}.$

12. Find the area between the curve $y^2 = \dfrac{x^3}{2a - x}$, its asymptote, and the X-axis.

114. Improper Integrals: Infinite Limits. In §113, the interval of integration was finite. In other words, neither of the limits of the integral

$$\int_a^b f(x)dx$$

was infinite.

The integral

$$\int_0^\infty \frac{dx}{x^2+a^2}$$

will be defined as

$$\lim_{b\to\infty} \int_0^b \frac{dx}{x^2+a^2}.$$

if this limit exists. Now

$$\lim_{b\to\infty}\int_0^b \frac{dx}{x^2+a^2} = \lim_{b\to\infty}\frac{1}{a}\tan^{-1}\frac{x}{a}\bigg|_0^b = \lim_{b\to\infty}\frac{1}{a}\tan^{-1}\frac{b}{a} = \frac{1}{a}\frac{\pi}{2}.$$

$\lim_{b\to\infty}\int_0^b \frac{dx}{x^2+a^2}$ represents graphically the limit of the area under the curve $y = \frac{1}{x^2+a^2}$ between the ordinates $x = 0$ and $x = b$ as b increases indefinitely.

Consider

$$\int_1^\infty \frac{dx}{x}.$$

$$\lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty}\log x\bigg|_1^b = \lim_{b\to\infty}\log b.$$

But $\log b$ increases without limit as b increases without limit. Hence $\int_1^\infty \frac{dx}{x}$ has no meaning.

Exercises

Evaluate the following integrals if they have a meaning:

1. $\displaystyle\int_0^\infty \frac{dx}{(1+x)^2}.$

2. $\displaystyle\int_0^\infty e^{-x}\,dx.$

3. $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^{\frac{3}{2}}}.$

4. $\displaystyle\int_2^\infty \frac{dx}{x\sqrt{x^2+4}}.$

5. Find the area between the curve, $y = \frac{8a^3}{x^2+4a^2}$, and the axis of x.

CHAPTER XIII

SOLID GEOMETRY

115. Coordinate Axes. Coordinate Planes. Just as the position of a point in a plane is given by two coordinates, for example, by its perpendicular distances from two mutually perpendicular coordinate axes, the position of a point in space is given by three coordinates, for example, by its perpendicular distances from three

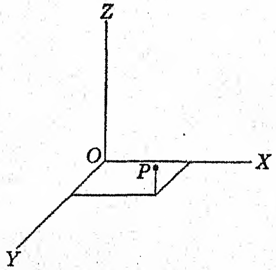

FIG. 71.

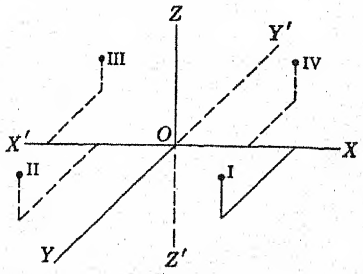

FIG. 72.

mutually perpendicular planes of reference, called the coordinate planes. Let the three coordinate planes be those represented in Fig. 71, *viz.*, XOY, called the XY-plane, YOZ, called the YZ-plane and ZOX, called the ZX-plane. Then the position of the point P whose perpendicular distances from the YZ-, ZX-, and XY-planes are 3, 2, and 1, respectively, is represented by the coordinates 3, 2, and 1. The lines of intersection of the planes of reference are called the axes. Thus $X'OX$, $Y'OY$, and $Z'OZ$, Fig. 72, are called the axes of x, y, and z, respectively. The coordinates of a point P measured parallel to these axes are known as its x, y, and z coordinates, respectively. Thus for the particular point P of Fig. 71, $x = 3$, $y = 2$, and $z = 1$. More briefly we say that the

point P is the point (3, 2, 1). In general, (x, y, z) is a point whose coordinates are x, y, and z. If these coordinates are given, the position of the point is determined, and if a point is given, these coordinates are determined.

The relation between a function of a single independent variable and its argument can be represented in a plane by a curve, the ordinates of which represent the values of the function corresponding to the respective values of the abscissas. Thus, $y = f(x)$ is represented by a curve. To an abscissa representing a given value of the argument there correspond one or more points on the curve whose ordinates represent the values of the function. In like manner a function of two independent variables x and y can be represented in space. Choose the system of coordinate planes of Fig. 71. Assign values to each of the independent variables x and y. These values fix a point in the XY-plane. At this point erect a perpendicular to the XY-plane, whose length z represents the value of the function corresponding to the given values of x and y. Thus a point P is determined. And for all values of x and y in a given region of the XY-plane there will, in general, correspond points in space. The locus of these points is a surface. The surface represents the relation between the function and its two independent arguments, just as a curve represents the relation between a function and its single argument.

Thus if $z = \pm\sqrt{25 - x^2 - y^2} = f(x, y)$, $\pm\sqrt{12}$ are the values of the function corresponding to the values $x = 2$ and $y = 3$. Then the points $(2, 3, 2\sqrt{3})$ and $(2, 3, -2\sqrt{3})$ lie on the surface $z = \pm\sqrt{25 - x^2 - y^2}$. If $x = -3$ and $y = 1$, $z = \pm\sqrt{15}$. The corresponding points on the surface are $(-3, 1, \sqrt{15})$ and $(-3, 1, -\sqrt{15})$.

The coordinate planes divide space into eight octants. Those above the XY-plane are numbered as shown in Fig. 72. The octant immediately below the first is the fifth, that below the second is the sixth, and so on. The points $(2, 3, 2\sqrt{3})$ and $(2, 3, -2\sqrt{3})$ lie in the first and fifth octants, respectively. The points $(-3, 1, \sqrt{15})$ and $(-3, 1, -\sqrt{15})$ lie in the second and sixth octants, respectively.

The locus of points satisfying the equation

$$z = \pm\sqrt{25 - x^2 - y^2} \tag{1}$$

is a sphere of radius 5, for this equation can be written in the form $x^2 + y^2 + z^2 = 25$, which states that for any point P on the surface (1), $OP = \sqrt{x^2 + y^2 + z^2} = 5$. The left member is the square of the distance, OP, of the point $P(x, y, z)$, from O, since OP is the diagonal of a rectangular parallelopiped whose edges are x, y, and z. If then the coordinates of P satisfy (1), this point is at a distance 5 from the origin. It lies on the sphere, of radius 5, whose center is at the origin.

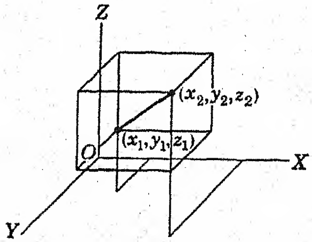

FIG. 73.

116. The Distance between Two Points. The student will show that the distance d between the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{1}$$

See Fig. 73. If the point (x_1, y_1, z_1) is the origin $(0, 0, 0)$, the expression for d becomes

$$\rho = \sqrt{x_2^2 + y_2^2 + z_2^2}. \tag{2}$$

Exercises

Find the distance between the following points:

1. (1, 2, 3) and (3, 5, 7).

2. (1, −2, 5) and (3, −2, −1).

3. (0, −3, 2) and (0, 0, 0).

4. (0, 0, 3) and (0, 2, 6).

5. (0, 0, −5) and (2, 0, 6).

6. (−3, 2, −1) and (0, 0, 0).

117. Direction Cosines of a Line. Let OL, Fig. 74, be any directed line segment passing through the origin. Let α, β, and γ be, respectively, the angles between this line and the positive directions of the X-, Y-, and Z-axes. These angles are called the direction angles of the line, and their cosines are called the *direction cosines of the line.* Let P, whose coordinates are x, y, and z, be any point on the line. Let $OP = \rho$. Then

$$x = \rho \cos \alpha,$$
$$y = \rho \cos \beta,$$
$$z = \rho \cos \gamma.$$

Squaring and adding we obtain

$$x^2 + y^2 + z^2 = \rho^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma).$$

Since

$$x^2 + y^2 + z^2 = \rho^2,$$
$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \quad (1)$$

The direction cosines of any line are defined as the direction cosines of a parallel line passing through the origin. Then *the sum of the squares of the direction cosines of any line is equal to unity.*

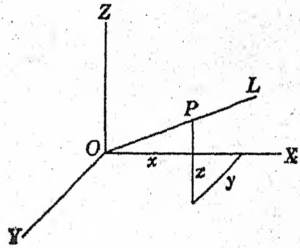

Fig. 74.

Exercises

Find the direction cosines of the lines passing through each of the following pairs of points.

1. (0, 0, 0) and (1, 1, 1).
2. (0, 0, 0) and (2, −3, 4).
3. (0, 0, 0) and (−1, 2, −3).
4. (1, 2, 3) and (5, 6, 7).
5. (−2, 3, −1) and (−3, −4, 3).

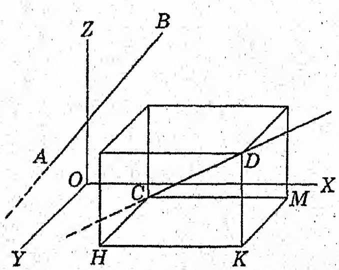

Fig. 75.

118. Angle between Two Lines. Let AB and CD, Fig. 75, be two lines, and let their direction cosines be $\cos \alpha_1$, $\cos \beta_1$, $\cos \gamma_1$, and $\cos \alpha_2$, $\cos \beta_2$, $\cos \gamma_2$, respectively. Denote the angle between the lines by θ. Let CM, MK, and KD be the edges of the parallelopiped formed by passing planes through C and D parallel to the coordinate planes. The projection of CD on AB is clearly equal to the sum of the projections of CM, MK, and KD on AB.

Hence

$$CD \cos\theta = CM \cos\alpha_1 + MK \cos\beta_1 + KD \cos\gamma_1.$$

Now

$$CM = CD \cos\alpha_2,$$
$$MK = CD \cos\beta_2,$$

and

$$KD = CD \cos\gamma_2.$$

Consequently,

$$CD \cos\theta = CD(\cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2).$$

Hence

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2. \quad (1)$$

Exercises

Find the cosine of the angle between the lines determined by the points of Exercises 1 and 2, 2 and 3, 3 and 4 of the preceding section.

119. The Normal Form of the Equation of a Plane. Let ABC, Fig. 76, be a plane. Let ON, the normal from O, meet the plane in N. Let the length of ON be p and let its direction angles be α, β, and γ. If ρ, α, β, and γ are given, the plane is determined.

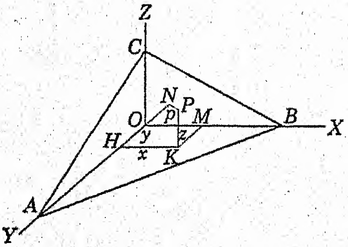

Fig. 76.

We wish to find the equation of the plane. Let P, with coordinates x, y, and z, be any point in the plane. The sum of the projections of $OM = x$, $MK = y$, and $KP = z$ upon ON is $ON = p$.

The projection of OM on ON is $x \cos\alpha$.
The projection of MK on ON is $y \cos\beta$.
The projection of KP on ON is $z \cos\gamma$.

Hence

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p. \tag{1}$$

If P does not lie in the plane ABC, the projection of the point P does not fall on N, and the coordinates of P do not satisfy (1). Hence the locus of a point satisfying (1) is a plane. Equation (1) is the normal form of the equation of the plane. p is taken to be positive. The algebraic signs of $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are determined by the octant into which ON extends.

Illustration 1. Find the equation of a plane for which $p = 2$, $\alpha = 60°$, $\beta = 45°$.

$$\cos \alpha = \tfrac{1}{2},$$
$$\cos \beta = \frac{1}{\sqrt{2}}.$$

Then by (1), § **117**,

$$\cos^2 \gamma = 1 - \tfrac{1}{4} - \tfrac{1}{2}.$$

Hence

$$\cos \gamma = \pm \tfrac{1}{2}.$$

The equation of the plane is

$$\frac{x}{2} + \frac{y}{\sqrt{2}} \pm \frac{z}{2} = 2.$$

There are thus two planes satisfying the conditions of the problem, one forming with the coordinate planes a tetrahedron in the first octant, the other a tetrahedron in the fifth octant.

Exercises

1. Find the equation of a plane if $\alpha = 60°$, $\beta = 135°$, $p = 2$, and if the normal ON extends into the eighth octant.

2. If $\alpha = 120°$, $\beta = 60°$, $p = 5$ and if the normal ON extends into the sixth octant.

120. The Equation Ax + By + Cz = D. The general equation of the first degree in x, y, and z is

$$Ax + By + Cz = D, \tag{1}$$

where A, B, C, and D are real constants. D may be considered positive, for if the constant term in the second member of an equation of the form (1) is not positive, it can be made so by dividing through by -1.

Divide (1) by $\sqrt{A^2 + B^2 + C^2}$ and obtain

$$\frac{A}{\sqrt{A^2 + B^2 + C^2}}x + \frac{B}{\sqrt{A^2 + B^2 + C^2}}y + \frac{C}{\sqrt{A^2 + B^2 + C^2}}z = \frac{D}{\sqrt{A^2 + B^2 + C^2}}. \qquad (2)$$

The coefficients of x, y, and z in equation (2) are the direction cosines of a line joining the origin to the point (A, B, C). Then equation (2) is in the form

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \qquad (3)$$

the normal form of the equation of a plane. Hence equation (1) from which (2) was derived represents a plane. Therefore every equation of the first degree in x, y, and z represents a plane.

Illustration 1. Put $3x - 2y - z = 6$ in the normal form. Divide by $\sqrt{A^2 + B^2 + C^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$ and obtain

$$\frac{3x}{\sqrt{14}} - \frac{2y}{\sqrt{14}} - \frac{z}{\sqrt{14}} = \frac{6}{\sqrt{14}}.$$

The plane is $\dfrac{6}{\sqrt{14}}$ units distant from the origin, and forms, with the coordinate planes, a tetrahedron in the eighth octant.

Exercises

Transform each of the following equations to the normal form, find the distance of each plane from the origin, and state in which octant it forms a tetrahedron with the coordinate planes.

1. $3x - 2y - z = 1$.

2. $x + y + z = -1$.

3. $x - 3y + 2z = 3$.

4. $x - 2y + 3z + 2 = 0$.

5. $2x - y - z - 1 = 0$.

6. $x + 2y = 6$.

7. $x - z = 4$.

8. $x = 2$.

9. $x = -1$.

10. $z = y$.

121. Intercept Form of the Equation of a Plane. The equation

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

is the equation of a plane, since it is of the first degree in x, y, and z. Its intercept on the X-axis, found by letting y and z equal zero, is a. Its intercepts on the Y- and Z-axes are b and c, respectively.

Illustration. Transform the equation $3x-2y-5z=4$ to the intercept form. Divide by 4 and obtain

$$\frac{x}{\frac{4}{3}}+\frac{y}{-2}+\frac{z}{-\frac{4}{5}}=1.$$

The intercepts on the X-, Y-, and Z-axes are $\frac{4}{3}$, -2, and $-\frac{4}{5}$, respectively.

Exercises

Transform each of the following equations to the intercept form:

1. $x+y+z=3$.
2. $2x-3y+4z=7$.
3. $2x+y-z+2=0$.
4. $2x+7y-3z=1$.
5. $x-y+3z=-\frac{1}{4}$.
6. $y-2x-3z=5$.

122. The Angle between Two Planes. The angle between two planes is the angle between the normals drawn to them from the origin. The cosine of the angle between the normals can be found by formula (1), **§118,** in which $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are the direction angles of the normals.

Illustration. Find the angle between the planes

$$x+y+z=1 \qquad (1)$$

and

$$2x+y+2z=3. \qquad (2)$$

Transform these equations to the normal form and obtain

$$\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{3}}=\frac{1}{\sqrt{3}} \qquad (3)$$

and

$$\frac{2x}{3}+\frac{y}{3}+\frac{2z}{3}=1. \qquad (4)$$

The direction cosines of the normals to the first and second planes are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$, and $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$, respectively. Then, if θ is the angle between the normals, formula (1), §**118,** gives

$$\cos\theta = \frac{2}{3\sqrt{3}} + \frac{1}{3\sqrt{3}} + \frac{2}{3\sqrt{3}} = \frac{5}{3\sqrt{3}},$$

from which $\theta = 74.5°$.

Exercises

Find the angle between the following pairs of planes:

1. $x - 3y + 2z = 6$ and $x - 2y + z = 1$.

2. $x - 2y + 3z = 2$ and $2x + y - 2z = 3$.

123. Parallel and Perpendicular Planes. If two planes are parallel, $\theta = 0$ and $\cos\theta = 1$. If they are perpendicular, $\theta = 90°$ and $\cos\theta = 0$.

Let

$$A_1x + B_1y + C_1z = D_1 \tag{5}$$

and

$$A_2x + B_2y + C_2z = D_2 \tag{6}$$

be the equations of two planes. After writing these equations in the normal form it is found that

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \tag{7}$$

If

$$A_1A_2 + B_1B_2 + C_1C_2 = 0, \tag{8}$$

$\cos\theta = 0$ and the planes (5) and (6) are perpendicular.

If the planes (5) and (6) are parallel, the corresponding coefficients must be equal or proportional, for then and only then will their normals be parallel.

Exercises

From the following equations pick out pairs of parallel planes and pairs of perpendicular planes.

1. $x + y + z = 6.$
2. $x - y - z = 2.$
3. $2x + 2y + 2z = 7.$
4. $3x - 2y - z = 8.$
5. $2x - 3y + z = 1.$

124. The Distance of a Point from a Plane. Let (x_1, y_1, z_1) be any point and let

$$Ax + By + Cz = D$$

be the equation of a plane. We shall find the distance of the point from the plane.

Now

$$Ax + By + Cz = K,$$

where K is any constant, is the equation of a plane parallel to the given plane (see §123). Let us choose K so that this plane shall pass through the given point (x_1, y_1, z_1). To do this substitute the coordinates of the point in the equation and solve for K. This gives

$$K = Ax_1 + By_1 + Cz_1.$$

Writing the equation of each plane in the normal form we have

$$\frac{Ax + By + Cz}{R} = \frac{D}{R}$$

and

$$\frac{Ax + By + Cz}{R} = \frac{K}{R} = \frac{Ax_1 + By_1 + Cz_1}{R},$$

where $R = \sqrt{A^2 + B^2 + C^2}$.

The given plane is $\frac{D}{R}$ units distant from the origin, and the plane through the point (x_1, y_1, z_1) is $\frac{Ax_1 + By_1 + Cz_1}{R}$ units distant from the origin. Then the distance d between the two planes, and hence the distance of the given point from the given plane is equal to the difference of these two distances, or

$$\mathbf{d} = \frac{\mathbf{Ax_1 + By_1 + Cz_1 - D}}{\sqrt{\mathbf{A^2 + B^2 + C^2}}}.$$

Illustration. Find the distance of the point (1, 2, −1) from the plane $3x - y + z + 7 = 0$.

$$d = \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}} = \frac{3 \cdot 1 - 1 \cdot 2 + 1 \cdot (-1) + 7}{\sqrt{3^2 + (-1)^2 + 1^2}}$$
$$= \frac{7}{\sqrt{11}}.$$

Exercises

In each of the following find the distance of the given point from the given plane:

1. $(3, 1, -2)$; $3x + y - 2z - 6 = 0$.
2. $(-1, 2, -3)$; $x - y - 2z + 1 = 0$.
3. $(0, 2, -3)$; $2x + 3y - 5z - 10 = 0$.

125. Symmetrical Form of the Equations of a Line. Let PP_1, Fig. 77, be a line passing through the given point P_1 (x_1, y_1, z_1), and having the direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$. In order to find the equations of the line, let P (x, y, z) be any point on the line and denote the distance PP_1 by d. Then

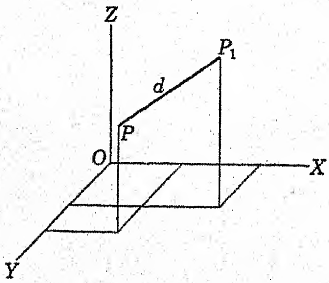

FIG. 77.

$$x - x_1 = d\cos\alpha,$$
$$y - y_1 = d\cos\beta,$$
$$z - z_1 = d\cos\gamma,$$

and therefore

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}. \quad (1)$$

These equations are known as the symmetric equations of the straight line.

Frequently a straight line is represented by the equations of two planes of which it is the intersection.

Illustration 1.

$$3x - y + 1 = 0, \quad (2)$$
$$5x - z = 3. \quad (3)$$

From these equations the symmetrical form of the equations can

readily be obtained. From (2) and (3) we obtain

$$x = \frac{y-1}{3} = \frac{z+3}{5},$$

or

$$\frac{x-0}{1} = \frac{y-1}{3} = \frac{z+3}{5}. \tag{4}$$

The denominators, 1, 3, and 5 of (4) are not the direction cosines of the line, but they are proportional to them. Upon dividing each by $\sqrt{35}$, the square root of the sum of their squares, they become the direction cosines. Then

$$\frac{x-0}{\frac{1}{\sqrt{35}}} = \frac{y-1}{\frac{3}{\sqrt{35}}} = \frac{z+3}{\frac{5}{\sqrt{35}}}$$

is the symmetrical form of the equations of the line.

The line, therefore, passes through the point $(0, 1, -3)$ and has the direction cosines given by the denominators in the preceding equations.

Illustration 2. Consider the line which is the intersection of the planes

$$\begin{aligned} 13x + 5y - 4z &= 40, \\ -13x + 10y - 2z &= 23. \end{aligned}$$

On eliminating x we obtain

$$5y - 2z = 21,$$

and on eliminating y we obtain

$$13x - 2z = 19.$$

From the last two equations we find

$$z = \frac{5y-21}{2} = \frac{13x-19}{2},$$

or

$$\frac{x - \frac{19}{13}}{\frac{2}{13}} = \frac{y - \frac{21}{5}}{\frac{2}{5}} = \frac{z-0}{1}.$$

These are the equations of a line which passes through the point $(\frac{19}{13}, \frac{21}{5}, 0)$ and whose direction cosines are proportional to $\frac{2}{13}$, $\frac{2}{5}$, and 1.

In *Illustration* 1, equation (2) represents a plane parallel to the Z-axis whose trace in the XY-plane is the line $3x - y + 1 = 0$. Equation (3) represents a plane parallel to the Y-axis whose trace in the ZX-plane is the line $5x - z = 3$.

In *Illustration* 2 the position of the two planes which intersect in the straight line is not so evident. By eliminating first x and then y, the equations of two planes passing through the same line are obtained, one of which is parallel to the X-axis and the other to the Y-axis.

Exercises

Put the equations of the following lines in the symmetrical form:

1. $x + 2y + 3z = 6,$
$x - y - z = 1.$

2. $x + y - z = 1,$
$x - 3y + 2z = 6.$

3. $x - y + 2z = 0,$
$x + 2y + 3z = 0.$

126. Surfaces of Revolution. Consider the surface whose equation is

$$y^2 + z^2 = 4x. \tag{1}$$

On letting $z = 0$, the trace of this surface in the XY-plane is found to be the parabola

$$y^2 = 4x. \tag{2}$$

Let

$$x = k, \tag{3}$$

where k is a constant. This is the equation of the plane $CPMN$, perpendicular to the X-axis in Fig. 78. Equations (1) and (3), considered as simultaneous equations, represent the curve of intersection of the surface (1) with the plane (3). If x is elimi-

nated between (1) and (3), there results

$$y^2 + z^2 = 4k. \tag{4}$$

The trace of the surface (1) in the plane (3) is, accordingly, the circle (4), whose center is at the point $(k, 0, 0)$ on the X-axis. In Fig. 78, this trace is the circle PMN. Its radius is $CP = 2\sqrt{k}$, the ordinate, corresponding to $x = k$, of the parabola OP. The surface (1) is, accordingly, the paraboloid of revolution generated by revolving the parabola $y^2 = 4x$ about the X-axis.

Similarly,

$$\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{16} = 1 \tag{5}$$

is the ellipsoid of revolution generated by revolving the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \tag{6}$$

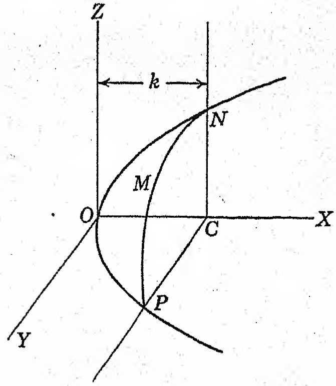

Fig. 78.

about the Y-axis. The trace of the surface in the XY-plane is the ellipse (6) and the trace in a plane, $y = k$, parallel to the XZ-plane, is a circle.

Exercises

Describe and sketch the following surfaces.

1. $x^2 + z^2 = 4y$.
2. $\dfrac{x^2}{9} + \dfrac{y^2}{4} + \dfrac{z^2}{4} = 1$.
3. $x^2 + y^2 + 2z = 0$.
4. $\dfrac{x^2}{16} - \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$.
5. $y^2 + z^2 = m^2x^2$.
6. $y = x^2 + z^2 - a^2$.
7. $\dfrac{x^2}{25} + \dfrac{y^2}{25} + \dfrac{z^2}{16} = 1$.
8. $x^2 + y^2 = 2z$.
9. $\dfrac{x^2}{16} + \dfrac{y^2}{16} - \dfrac{z^2}{9} = 1$.
10. $\dfrac{x^2}{25} - \dfrac{y^2}{9} - \dfrac{z^2}{9} = 1$.

127. Cylindrical Surfaces. The equation

$$x^2 + y^2 = a^2$$

is independent of z. Consequently, the traces of the surface which it represents are the same in all planes parallel to the XY-plane. They are circles of radius a whose centers lie on the Z-axis. The surface is, therefore, a right circular cylinder of radius a, whose axis is the Z-axis, and whose elements are lines parallel to the Z-axis. The surface may be regarded as generated by a line moving parallel to the Z-axis and passing through points of the circle, $x^2 + y^2 = a^2$, in the XY-plane. Clearly, the same relation will hold between x and y at a point on the surface and in any plane, $z = k$, as that which holds at the point lying on the same element in the XY-plane.

In general, a cylindrical surface is a surface generated by a line moving parallel to a fixed line and passing through the points of a given curve.

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represents an elliptical cylinder whose elements are parallel to the Z-axis.

The equation

$$y^2 + z^2 = a^2$$

represents a right circular cylinder whose elements are parallel to the X-axis.

The equation

$$z^2 = 4x$$

represents a parabolic cylinder whose elements are parallel to the Y-axis.

The plane

$$x - 4y + 3 = 0$$

may be regarded as a cylindrical surface whose elements are parallel to the Z-axis and which pass through the line

$$y = \frac{x}{4} + \frac{3}{4}$$

in the XY-plane.

In general, an equation in which one of the three letters x, y, or z is absent represents a cylindrical surface whose elements are parallel to the axis corresponding to the letter which does not appear in the equation.

Exercises

Describe the surfaces represented by the following equations:

1. $x^2 + y^2 = 16.$
2. $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1.$
3. $x^2 - y^2 = 1.$
4. $z^2 + y^2 = 25.$
5. $z^2 - x^2 = 25.$
6. $x + 3y = 10.$
7. $x^2 - y^2 = 0.$
8. $xy = 1.$
9. $xz = 2.$
10. $(x - 3)(x + z) = 0.$
11. $y^2 = 4x.$
12. $y^2 + z^2 = 2ay.$
13. $x^2 + y^2 = 10x.$

128. Quadric Surfaces. Any equation of the second degree between x, y, and z, of which

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Kz + L = 0 \tag{1}$$

is the general form, represents a surface which is called a *quadric surface*, or *conicoid*.

By a suitable rotation and translation of the axes, the equation of any quadric surface can be put in one of the following forms:

$$\pm \frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1, \tag{2}$$

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 0, \tag{3}$$

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \pm 2cz. \tag{4}$$

The particular form assumed by the equation depends upon the values of the coefficients in (1).

Consider the quadric surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{5}$$

The equation of the trace of this surface in a plane $x = k$, parallel to the YZ-plane, is the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}. \tag{6}$$

This equation may be written in the form

$$\frac{y^2}{\left(\frac{b\sqrt{a^2 - k^2}}{a}\right)^2} + \frac{z^2}{\left(\frac{c\sqrt{a^2 - k^2}}{a}\right)^2} = 1. \tag{7}$$

The axes grow shorter as k increases in numerical value from 0 to a. When $k = \pm a$, the elliptical section reduces to a point. When $|k| > a$, the lengths of the axes of the ellipse become imaginary, *i.e.*, the plane $x = k$, ($|k| > a$), does not meet the surface (5) in real points. Hence the surface is included between the planes $x = \pm a$.

The above discussion shows that the surface represented by the equation (5) is included between the planes $x = \pm a$; is symmetrical with respect to the YZ-plane; and has elliptical sections made by planes perpendicular to the axis of x. These sections grow smaller as the cutting plane is moved away from the YZ-plane and at a distance $\pm a$ reduce to a point.

In a similar manner the sections made by the planes, $y = k$ and $z = k$, parallel to the XZ- and XY-planes, respectively, are found to be ellipses. a, b, and c are called the semi-axes of the ellipsoid. The surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{8}$$

is called an hyperboloid of one sheet, or of one nappe.

The trace of this surface in a plane $z = k$ parallel to the XY-plane is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}$$

whose semi-axes are

$$\frac{a}{c}\sqrt{c^2 + k^2} \quad \text{and} \quad \frac{b}{c}\sqrt{c^2 + k^2}.$$

These semi-axes increase in length as k increases in numerical value. They are shortest when $k = 0$. The surface represented by equation (8) is symmetrical with respect to the XY-plane, and every section parallel to this plane is an ellipse. The smallest elliptical section is that made by the XY-plane.

The trace of the surface in the XZ-plane obtained by letting $y = 0$ is the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1.$$

With the information at hand it is possible to make a sketch of the surface (Fig. 79).

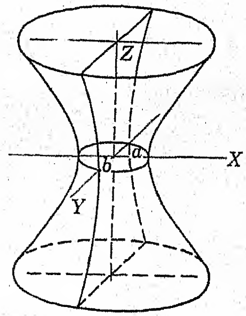

Fig. 79.

The sections made by planes $x = k$ and $y = k$ will be found to be hyperbolas.

Exercises

The student will discuss the following surfaces and make sketches of them:

1. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$, the hyperboloid of two sheets.

2. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2cz$, the hyperbolic paraboloid.

3. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2cz$, the elliptic paraboloid.

129. Partial Derivatives. Let $z = f(x, y)$ be a function of two independent variables, x and y. When x takes on an increment Δx, while y remains fixed z takes on an increment which we shall denote by $\Delta_x z$. When y takes on an increment, Δy, while x remains fixed, z takes on an increment which we shall denote by $\Delta_y z$.

For example, if a gas be enclosed in a cylinder with a movable piston, the volume v of the gas is a function of the temperature T and of the pressure p which can be varied by varying the pressure on the piston. If the temperature alone be changed, the volume will take on a certain increment $\Delta_T v$. If the pressure alone be changed, the volume will take on the increment $\Delta_p v$.

If $z = f(x, y)$ be represented by a surface, Fig. 80, the increment of z obtained by giving x an increment, while y remains constant, is the increment in z measured to the curve cut out by a plane $y = k$, a constant. Thus $\Delta_x z = KR$; similarly, $\Delta_y z = HQ$.

The limit of the quotient $\frac{\Delta_x z}{\Delta x}$ as Δx approaches zero is called the partial derivative of z with respect to x. It is denoted by the symbol $\frac{\partial z}{\partial x}$. Then

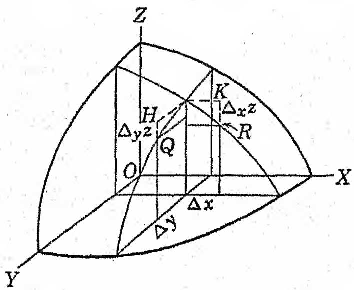

FIG. 80.

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x}.$$

It is evidently calculated from $z = f(x, y)$ by the ordinary rules of differentiation, y being treated as a constant. Thus if $z = x^2 y$,

$$\frac{\partial z}{\partial x} = 2xy.$$

Geometrically, $\frac{\partial z}{\partial x}$ represents the slope of the tangent at the point (x, y, z) to the curve cut from the surface by the plane through this point parallel to the XZ-plane.

Similarly,

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y}$$

and it is calculated by differentiating $z = f(x, y)$, treating x as a constant. Geometrically, it represents the slope of the tangent at the point (x, y, z) to the curve cut from the surface by the plane through this point, parallel to the YZ-plane. If $z = x^2 y$,

$$\frac{\partial z}{\partial y} = x^2.$$

Illustration 1. If $z = \sin \frac{x}{y}$,

$$\frac{\partial z}{\partial x} = \cos \frac{x}{y} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y} \cos \frac{x}{y}$$

and

$$\frac{\partial z}{\partial y} = \cos \frac{x}{y} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{x}{y^2} \cos \frac{x}{y}.$$

130. Partial Derivatives of Higher Order. If z is differentiated twice with respect to x, y being treated as a constant, the derivative obtained is called the second partial derivative of z with respect to x. It is denoted by the symbol $\frac{\partial^2 z}{\partial x^2}$. Similarly, the second partial derivative of z with respect to y is denoted by the symbol $\frac{\partial^2 z}{\partial y^2}$.

If z is differentiated first with respect to x, y being treated as a constant, and then with respect to y, x being treated as a constant the result is denoted by the symbol $\frac{\partial^2 z}{\partial y\, \partial x}$. If the differentiation takes place in the reverse order, the result is denoted by the symbol $\frac{\partial^2 z}{\partial x\, \partial y}$. The first is read "the second partial derivative of z with respect to x and y"; the second, "the second partial derivative of z with respect to y and x." In the case of functions usually occurring in physics and engineering, *viz.*, functions which are continuous and which have continuous first and second partial derivatives, $\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y}$. The order of differentiation is immaterial.

Illustration 1. $z = x^2 y$.

$$\frac{\partial z}{\partial x} = 2xy, \qquad \frac{\partial^2 z}{\partial x^2} = 2y, \qquad \frac{\partial^2 z}{\partial y\, \partial x} = 2x.$$

$$\frac{\partial z}{\partial y} = x^2, \qquad \frac{\partial^2 z}{\partial y^2} = 0, \qquad \frac{\partial^2 z}{\partial x\, \partial y} = 2x.$$

In this case

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}.$$

Illustration 2. $z = \sin \dfrac{x}{y}$.

$$\frac{\partial z}{\partial x} = \frac{1}{y}\cos\frac{x}{y}.$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{y^2}\sin\frac{x}{y}.$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{1}{y}\left(-\sin\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) - \frac{1}{y^2}\cos\frac{x}{y}$$

$$= \frac{1}{y^3}\left(x\sin\frac{x}{y} - y\cos\frac{x}{y}\right).$$

$$\frac{\partial z}{\partial y} = -\frac{x}{y^2}\cos\frac{x}{y}.$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x}{y^3}\cos\frac{x}{y} - \frac{x^2}{y^4}\sin\frac{x}{y}.$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{x}{y^2}\left(\sin\frac{x}{y}\right)\left(\frac{1}{y}\right) - \frac{1}{y^2}\cos\frac{x}{y}$$

$$= \frac{1}{y^3}\left(x\sin\frac{x}{y} - y\cos\frac{x}{y}\right).$$

Here, again, we notice that

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y}.$$

Exercises

1. Find $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$, $\dfrac{\partial^2 z}{\partial x\,\partial y}$, and $\dfrac{\partial^2 z}{\partial y\,\partial x}$, for each of the functions:

(*a*) $z = \dfrac{x}{y}$. (*c*) $z = x^3y$. (*e*) $z = e^x \sin y$.

(*b*) $z = xy^2$. (*d*) $z = \sin xy$.

2. Find $\dfrac{\partial^2 z}{\partial y\,\partial x}$ and $\dfrac{\partial^2 z}{\partial x\,\partial y}$ for each of the following functions:

(*a*) $z = x^2y$. (*c*) $z = x\cos y$. (*e*) $z = e^y \sin x$.

(*b*) $z = x\sin^{-1} y$. (*d*) $z = y\log x$. (*f*) $z = y\tan^{-1} x$.

It is seen that $\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y}$ in all of these cases.

In the above discussion z was considered to be a function of two independent variables only. The notion of partial derivatives can, however, be extended to functions of three or more variables.

Illustration 3. If $z = x^2yt$,

$$\frac{\partial z}{\partial x} = 2xyt,$$

$$\frac{\partial z}{\partial y} = x^2t,$$

$$\frac{\partial z}{\partial t} = x^2y,$$

$$\frac{\partial^2 z}{\partial x\, \partial t} = 2xy,$$

and

$$\frac{\partial^3 z}{\partial t\, \partial y\, \partial x} = 2x.$$

CHAPTER XIV

SUCCESSIVE INTEGRATION: CENTER OF GRAVITY: MOMENT OF INERTIA

131. Successive Integration. The symbol

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dy\,dx \tag{1}$$

indicates that two integrations are to be performed in succession, first with respect to y, treating x as a constant, and then with respect to x. The expression (1) will be referred to as a double integral.[1] The process of calculating a double integral is shown in the following example:

$$\begin{aligned}\int_0^1 \int_{x^2}^{x} (x^2 + y^2)dy\,dx &= \int_0^1 \left(x^2y + \frac{y^3}{3}\right)\Bigg|_{y=x^2}^{y=x} dx \\ &= \int_0^1 \left(\frac{4x^3}{3} - x^4 - \frac{x^6}{3}\right)dx = \left(\frac{x^4}{3} - \frac{x^5}{5} - \frac{x^7}{21}\right)\Bigg|_0^1 \\ &= \frac{3}{35}.\end{aligned}$$

A triple integral (see Exercise 10 below) is calculated in a similar way.

If in a double integral dy is written before dx, the integration with respect to y is to be performed first.[2]

[1] The student who pursues his study of mathematics further will have occasion to use this term in a somewhat different sense.

[2] Usage varies on this point. The student will have to observe in every case the convention adopted in the book he is reading.

Exercises

1. $\int_0^1 \int_0^x x^2y\,dy\,dx.$

2. $\int_0^1 \int_2^5 xy\,dy\,dx.$

3. $\int_0^1 \int_{x^2}^{\sqrt{x}} dy\,dx.$

4. $\int_0^1 \int_{y^2}^{\sqrt{y}} x\,dx\,dy.$

5. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)dy\,dx.$

6. $\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\,dy\,dx.$

7. $\int_{-1}^2 \int_{x^2}^{x+2} dy\,dx.$

8. $\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dy\,dx.$

9. $\int_0^\pi \int_0^{5(1-\cos\theta)} \rho\,d\rho\,d\theta.$

10. $\int_0^2 \int_1^3 \int_1^2 xy^2z\,dz\,dy\,dx.$

11. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx.$

12. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dy\,dx.$

HINT. To perform the integration with respect to y in Exercise 12, let $\sqrt{a^2 - x^2} = b$ and make use of the result of *Illustration* 3, §105.

132. Geometrical Interpretation of the Definite Double Integral. Volume by Double Integration. Every definite single integral, $\int_a^b f(x)dx$, can be interpreted as representing an area (see §65). It will now be shown that every definite double integral can be interpreted as a volume. Consider

$$\int_b^a \int_{g_1(x)}^{g_2(x)} f(x, y)dy\,dx. \tag{1}$$

In Fig. 81, $EFGL$ represents the surface $z = f(x, y)$. $ABB'A'$ and $CDD'C'$ are the cylindrical surfaces, $y = g_1(x)$ and $y = g_2(x)$, respectively. $ADD'A'$ and $BCC'B'$ are the planes, $x = a$ and $x = b$, respectively. These planes and surfaces together with the plane $z = 0$ bound the solid BD'.

Divide this solid into vertical columns by passing planes parallel to the ZX- and ZY-planes, respectively. To avoid confusion in the figure only a part of the lines are drawn. A typical

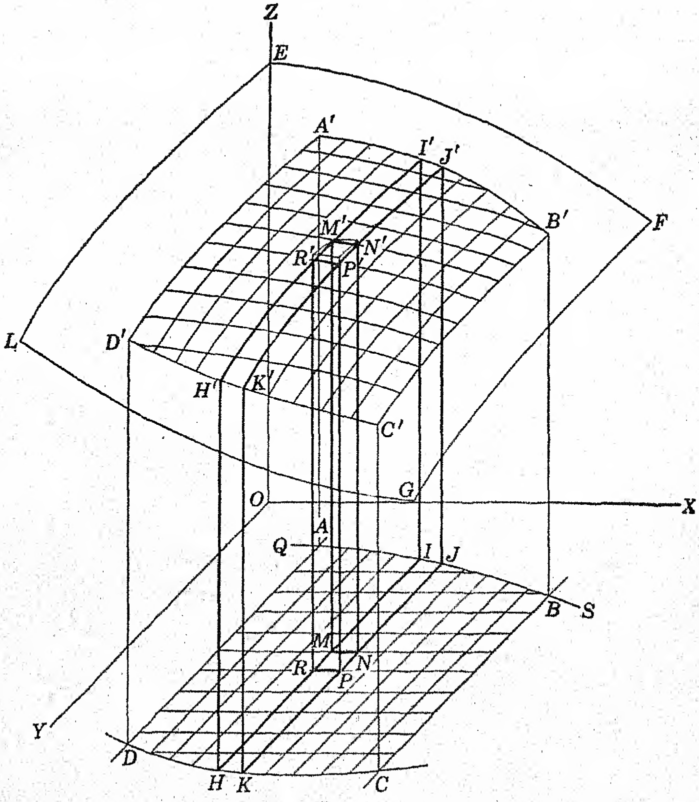

Fig. 81.

column is the one standing on the base $MNPR$. The approximate volume of this column is

$$z\,\Delta y\,\Delta x = f(x, y)\Delta y\,\Delta x,$$

viz., the volume of a right prism having the base $MNPR = \Delta y\,\Delta x$

and the altitude $MM' = z = f(x, y)$, M being the point $(x, y, 0)$. The sum

$$\sum_{y=g_1(x)}^{y=g_2(x)} f(x, y)\Delta y\, \Delta x,$$

x and Δx considered constant, represents approximately the sum of the columns included in the slab JH'. And

$$\left[\lim_{\Delta y \to 0} \sum_{y=g_1(x)}^{y=g_2(x)} f(x, y)\Delta y\right]\Delta x = \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\right]\Delta x$$

represents approximately the volume of the slab JH'. The definite integral within the brackets represents the area of the plane section $IHH'I'$. Δx is the thickness of the slab having this area as its base. The expression,

$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b}\left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\right]\Delta x = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx,$$

is the limit, as Δx approaches zero, of the sum of all such slabs included between the planes $x = a$ and $x = b$. That is, it represents the volume of the solid bounded by the plane $z = 0$, the surface $z = f(x, y)$, the cylinders $y = g_1(x)$, $y = g_2(x)$ and the planes $x = a$, $x = b$.

Illustration. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ (see Fig. 82).

The volume lying in the first octant will be found and multiplied by 8.

The volume of the column standing on the base $MNPR$ is

$$z\,dy\,dx = \sqrt{a^2 - x^2 - y^2}\,dy\,dx.$$

The summation (integration with respect to y) of the volumes of these columns for a fixed x, from $y = 0$, the trace of the ZX-plane in the XY-plane, to $y = \sqrt{a^2 - x^2}$, the trace of the sphere

in the XY-plane, gives

$$\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx = \frac{\pi(a^2 - x^2)}{4} dx,$$

the volume, expressed as a function of x, of the slab included between the planes $x = x$ and $x = x + dx$. The limits 0 and $\sqrt{a^2 - x^2}$ are the values of y at the points H and K, respectively. They replace the limits $g_1(x)$ and $g_2(x)$ of the general discussion.

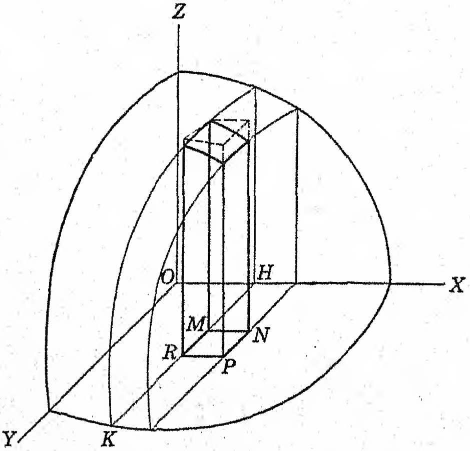

FIG. 82.

The summation (integration with respect to x) of all the slabs between $x = 0$ and $x = a$ gives the volume of the portion of the sphere lying in the first octant, *viz.*,

$$\frac{V}{8} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx$$

$$= \frac{\pi}{4}\int_0^a (a^2 - x^2)dx = \frac{\pi a^3}{6}.$$

Then

$$V = \tfrac{4}{3}\pi a^3.$$

It is interesting to notice that the result of the first integration gives as the element of the integration with respect to x the quantity $\frac{\pi}{4}(a^2 - x^2)dx$, which is identical with the element of integration, $\frac{\pi y^2}{4}\,dx$, of the single definite integral representing the same volume.

Exercises

Find by double integration the volume of:

1. The tetrahedron enclosed by the plane $x + y + z = 2$ and the coordinate planes.

2. One of the two wedges cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = mx$.

3. The solid bounded by the plane $z = 0$, the surface $z = x^2 + y^2 + 2$, and the surface $x^2 + y^2 = 4$.

4. The solid bounded by $y^2 + z^2 = 4x$ and the plane $x = 5$.

5. The solid bounded by the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.

6. The solid bounded by the cylinders $y = x^2$ and $y^2 = x$, and the planes $z = 0$ and $z = 1$.

7. The solid bounded by $z = 4 - x^2 - y^2$ and the plane $z = 0$.

8. The solid bounded by the surfaces $x^2 + y^2 = 4$, $z - x = 3$, and $z = 0$.

9. The solid bounded by the surfaces $x^2 + y^2 = az$, $x^2 + y^2 = 2ax$, and $z = 0$.

10. The solid bounded by the surfaces

$$x^2 + y^2 = 9,\; x + y + 5 = z,$$

and $z = 0$.

11. The solid bounded by the surfaces

$$x^2 + y^2 = 4,\; z = x + y + 3,$$

and $z = 0$.

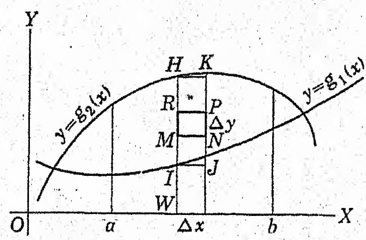

FIG. 83.

133. Area. Double Integration. Rectangular Coordinates. It is required to find by double integration the area bounded by the curves $y = g_1(x)$, $y = g_2(x)$ and the lines $x = a$ and $x = b$ (see Fig. 83). Subdivide the area in question by drawing lines parallel

to the X- and Y-axes at intervals Δy and Δx, respectively. The area of the rectangular strip $IJKH$ is given by the expression,

$$\lim_{\Delta y \to 0} \sum_{y=g_1(x)}^{y=g_2(x)} \Delta y\, \Delta x = \left[\lim_{\Delta y \to 0} \sum_{y=g_1(x)}^{y=g_2(x)} \Delta y \right] \Delta x = \left[\int_{g_1(x)}^{g_2(x)} dy \right] \Delta x,$$

x and Δx being constant. The integral within the brackets is equal to $g_2(x) - g_1(x)$, the length of the line IH. Δx is the width IJ of the rectangle $IJKH$. The area sought is the limit of the sum of rectangular strips, such as $IJKH$, as Δx approaches zero. That is,

$$A = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \left[\int_{g_1(x)}^{g_2(x)} dy \right] \Delta x = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx.$$

The first integration, that with respect to y between the limits $g_1(x)$ and $g_2(x)$, may be said to sum up the rectangles of area $\Delta y\, \Delta x$ in a strip of width Δx, parallel to the Y-axis. The second integration, that with respect to x, calculates the limit of the sum of these strips and gives the area sought.

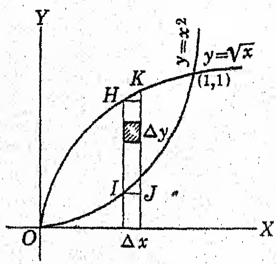

FIG. 84.

Illustration 1. Find by double integration the area between the parabolas $y^2 = x$ and $y = x^2$ (see Fig. 84).

$$A = \int_0^1 \int_{x^2}^{\sqrt{x}} dy\, dx = \frac{1}{3}.$$

Illustration 2. Find by double integration the area enclosed by $y^2 = x$ and $y = x - 2$. Here it is advantageous to integrate first with respect to x.

$$A = \int_{-1}^{2} \int_{y^2}^{y+2} dx\, dy.$$

In this case the integration with respect to x gives the area of a

horizontal strip of width Δy. The second integration, that with respect to y, calculates the limit of the sum of these strips. If we desire to integrate first with respect to y, two double integrals are necessary, for to the left of the dotted line in Fig. 85 the elementary rectangle parallel to the Y-axis extends from $y = -\sqrt{x}$ to $y = \sqrt{x}$. To the right of the dotted line, it extends from the line $y = x - 2$ to the parabola $y = \sqrt{x}$. Thus

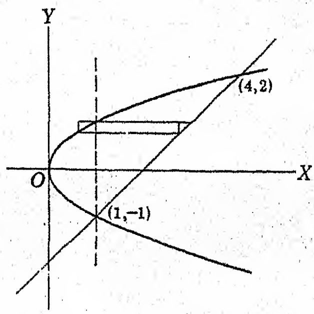

FIG. 85.

$$A = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} dy\,dx.$$

Exercises

Find by double integration:

1. The area between $y = x$ and $y = x^2$.
2. The area between $y^2 = x$ and $y = 2 - x$.
3. The area between $y^2 = 2(2 - x)$ and $y = 2 - x$.
4. The smaller of the areas between $y = 2 - x$ and $x^2 + y^2 = 4$.
5. The area enclosed by $y = 4 - x^2$ and $y = 4 - 2x$.
6. The area enclosed by $y^2 = 2x$ and $y = 3 - x$.
7. The smaller of the areas between $y^2 = 2ax - x^2$ and $y = x$.
8. The area between $y = x^2 - 1$ and $y = 1 - x$.
9. The area between $y^2 = 5 - x$ and $y = x + 1$.
10. The area between $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ and $y = a - x$.
11. The area bounded by $y^2 = 4(x + 1)$ and $y^2 + 2(x - 2) = 0$.
12. The area bounded by $y = 3 - x^2 - x$ and $y = 1 - 2x$.
13. The area bounded by $y = 4 + 2x - x^2$ and $y = 4 - 2x$.
14. The area bounded by $y^2 - 4y - x = 6$ and $x + y + 2 = 0$.
15. The area bounded by $2y^2 = x^3$ and $y^2 + 2x = 8$.
16. The area bounded by $y^2 = x$ and $3y = x + 2$.
17. The area bounded by $2(x - 1)^2 = (y - 1)^3$ and $x^2 - 2x + 2y = 9$.

134. Mean Distance and Mean Square of the Distance of an Area from a Line. Consider the area between the curves $y =$

$g_1(x)$, $y = g_2(x)$, and the lines $x = a$ and $x = b$ (see Fig. 86). The mean distance,[1] $\bar{x}$, of this area from the Y-axis is found by assigning to each x measured to an element of area, $\Delta y \, \Delta x$, a weight equal to that area and calculating the limit of the weighted mean of x so determined, as Δy and Δx approach zero. Hence $\bar{x}$ is the limit of the sum of expressions of the form $x \Delta y \, \Delta x$ divided by the limit of

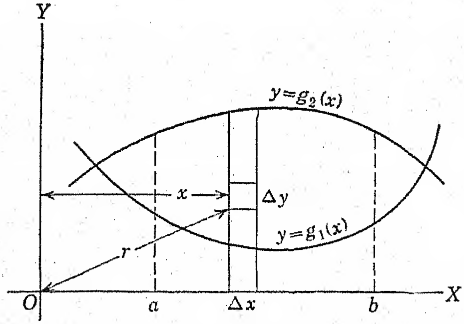

FIG. 86.

the sum of expressions of the form $\Delta y \, \Delta x$, *i.e.*, by the whole area. Thus

$$\bar{x} = \frac{\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} x \, dy \, dx}{\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} dy \, dx}.$$

This can be written briefly in the form

$$\bar{x} = \frac{\displaystyle\int x \, dA}{\displaystyle\int dA}.$$

In like manner the mean square of the distance of the area in question from the Y-axis is given by

[1] See *Illustration* 3, §77, where the mean distance of an area from a line is calculated without using double integrals.

$$\overline{x^2} = \frac{\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} x^2\,dy\,dx}{\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx}.$$

Briefly,

$$\overline{x^2} = \frac{\displaystyle\int x^2 dA}{\displaystyle\int dA}.$$

The mean square of the distance of the same area from a line through the origin perpendicular to the XY-plane is given by

$$\overline{r^2} = \frac{\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} (x^2 + y^2)dy\,dx.}{\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx}.$$

Frequently, the double integrals written above can be replaced by single integrals by choosing as element a strip parallel to the line from which the distance is measured.

Exercises

1. Find by double integration the mean distance from the Y-axis of the area lying in the first quadrant and bounded by the circle $x^2 + y^2 = a^2$.

2. Find the mean square of the distance of the area of Exercise 1 from the Y-axis.

3. Find the mean square of the distance of the area of Exercise 1 from a line through the origin perpendicular to the XY-plane.

4 Find the mean distance from the Y-axis of the area bounded by $y^2 = x$ and $x = 2$.

5. Find the mean square of the distance of the area of Exercise 4 from the X-axis.

6. Find the mean square of the distance of the area of Exercise 4 from a line through the origin perpendicular to the XY-plane.

7. Find the mean distance from the X- and Y-axes, respectively, of the area of the triangle enclosed by the lines $y = \frac{x}{2}$, $y = 0$, and $x = a$.

8. Find the mean square of the distance from the Y-axis of the area of the rectangle enclosed by the lines $x = 0$, $x = a$, $y = 0$, $y = b$.

9. Find the mean distance from the X-axis of the area between $y^2 = x$ and $y = x$.

10. Find the mean distance from the Y-axis of the area enclosed by $2y^2 = x^3$ and $y^2 + 2x = 8$.

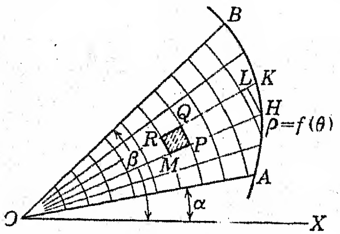

FIG. 87.

135. Area: Polar Coordinates. Let it be required to find, by double integration, the area between the radii $\theta = \alpha$, and $\theta = \beta$, and the curve $\rho = f(\theta)$. Divid the area as shown in Fig. 87, the radii making an angle of $\Delta\theta$ with each other and the radii of the concentric circles differing by $\Delta\rho$. The area of $MPQR$ is equal to

$$\tfrac{1}{2}(\rho + \Delta\rho)^2\,\Delta\theta - \tfrac{1}{2}\rho^2\,\Delta\theta = \rho\,\Delta\rho\,\Delta\theta + \tfrac{1}{2}(\Delta\rho)^2\,\Delta\theta.$$

As $\Delta\rho$ approaches zero,

$$\lim_{\Delta\rho\to 0}\frac{\rho\,\Delta\rho\,\Delta\theta + \frac{1}{2}(\Delta\rho)^2\,\Delta\theta}{\rho\,\Delta\rho\,\Delta\theta} = 1.$$

Hence

$$\lim_{\Delta\rho\to 0}\sum_{\rho=0}^{\rho=f(\theta)}(\rho\,\Delta\rho\,\Delta\theta + \tfrac{1}{2}\Delta\rho^2\Delta\theta) = \lim_{\Delta\rho\to 0}\sum_{\rho=0}^{\rho=f(\theta)}\rho\,\Delta\rho\,\Delta\theta = \int_{\rho=0}^{\rho=f(\theta)}\rho\,d\rho\,\Delta\theta.$$

This sum represents the area of the sector OHL. The total area sought is the limit of the sum of these sectors as $\Delta\theta$ approaches zero, *i.e.*,

$$A = \lim_{\Delta\theta\to 0}\sum_{\theta=\alpha}^{\theta=\beta}\Delta\theta\int_{\rho=0}^{\rho=f(\theta)}\rho\,d\rho = \int_{\alpha}^{\beta}\int_{0}^{f(\theta)}\rho\,d\rho\,d\theta.$$

Illustration 1. Find the area of the circle $\rho = 10 \cos \theta$, Fig. 88. The area bounded by the semicircle above the initial line will be

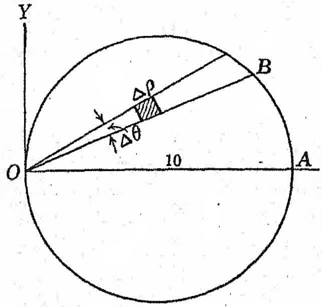

FIG. 88.

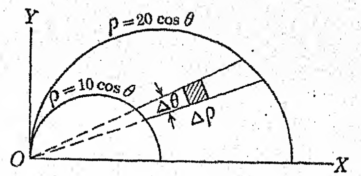

FIG. 89.

found and multiplied by 2.

$$A = 2\int_0^{\frac{\pi}{2}}\int_0^{10\cos\theta} \rho\, d\rho\, d\theta.$$

Show that

$$A = 2\int_0^{10}\int_0^{\cos^{-1}\frac{\rho}{10}} \rho\, d\theta\, d\rho.$$

Illustration 2. Find by double integration the area between $\rho = 10 \cos \theta$ and $\rho = 20 \cos \theta$ (see Fig. 89).

$$A = 2\int_0^{\frac{\pi}{2}}\int_{10\cos\theta}^{20\cos\theta} \rho\, d\rho\, d\theta.$$

Exercises

Find by double integration:

1. The area between $\rho = 2 \sin \theta$ and $\rho = 4 \sin \theta$.
2. The area of one loop of $\rho^2 = 9 \cos 2\theta$.
3. The area enclosed by $\rho = 2(1 + \cos \theta)$.
4. The area outside $\rho = a(1 + \cos \theta)$ and inside $\rho = 3a \cos \theta$.
5. The area outside $\rho = 3$ and inside $\rho = 6 \cos \theta$.

6. The area outside $\rho = 4$ and inside $\rho = 3 + 2\cos\theta$.

7. The area of the smaller loop of $\rho = a\cos\dfrac{\theta}{2}$.

8. The area of one loop of $\rho = 3\sin^2\theta$.

9. The area of the loop of $\rho = 4 + 2\sec\theta$.

10. The smaller of the areas enclosed by $\rho = 2\sec\left(\theta - \dfrac{\pi}{4}\right)$ and $\rho = 4$.

136. Volume of a Solid: Triple Integration. Let the solid of Fig. 81 be further subdivided by planes parallel to the XY-plane, forming a large number of rectangular parallelopipeds of volume $\Delta z\ \Delta y\ \Delta x$. By a familiar process of reasoning, it follows that the volume of the solid is expressed by

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_0^{f(x,y)} dz\ dy\ dx.$$

The integration with respect to z calculates the limit of the sum of the volumes of the parallelopipeds in the vertical column standing on a base $\Delta y\,\Delta x$, such as $MNPR$. The next integration calculates the limit of the sum of the columns in a slab of thickness Δx parallel to the ZY-plane. The last integration calculates the limit of the sum of the slabs.

Illustration 1. Find by triple integration the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ (see Fig. 90).

$$V = 8\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \int_0^{\frac{c}{b}\sqrt{\frac{b^2}{a^2}(a^2-x^2)-y^2}} dz\ dy\ dx.$$

The calculation of the integral is left as an exercise.

Illustration 2. Find by triple integration the volume of the solid bounded by the cylinder $x^2 + y^2 = 2ax$, the plane $z = 0$, and the paraboloid of revolution $x^2 + y^2 = 4az$. Write the element of volume, $dz\ dy\ dx$. The integration with respect to z between the limits 0 and $\dfrac{x^2 + y^2}{4a}$ gives the volume of the typical vertical column of base $dy\ dx$, extending from the point (x, y) in

the plane $z = 0$ to the surface of the paraboloid, $z = \dfrac{x^2 + y^2}{4a}$. Next, x being kept fixed, these columns are summed into a typical slab by integrating with respect to y from the X-axis, $y = 0$, to $y = \sqrt{2ax - x^2}$, the trace of the cylinder in the XY-plane. Finally, the integration with respect to x from $x = 0$ to $x = 2a$

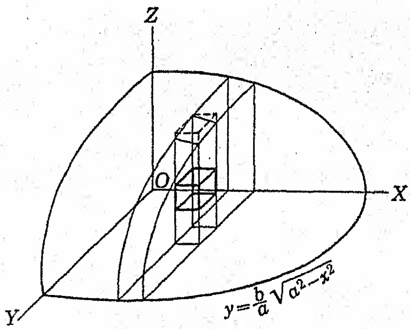

FIG. 90.

gives one-half of the total volume sought, *viz.*, that lying in the first octant.

$$V = 2\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\int_0^{\frac{x^2+y^2}{4a}} dz\, dy\, dx.$$

The student will perform the integration.

Exercises

Find by triple integration:

1. The volume in the first octant bounded by the coordinate planes and the plane $x + 3y + 2z = 6$.

2. The volume of one of the two wedges cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = mx$.

3. The volume of the cap cut from $y^2 + z^2 = 4x$ by the plane $z = x$.

4. The volume enclosed by the cylinder, $x^2 + y^2 = 9$, and the planes, $z = 5 - x$ and $z = 0$.

5. The volume enclosed by $y^2 + 2z^2 = 4x - 8$, $y^2 + z^2 = 4$, and the plane $x = 0$.

The expression (4) for $\bar{x}$, the abscissa of the center of gravity of the mass of a plate of uniform density, is identical with the expression for the mean distance of the area of the plate from the Y-axis (see **§134**), since in the expression (4) there is no reference to mass. We shall refer to $(\bar{x}, \bar{y})$ as the center of gravity of the area. It is identical with the center of gravity of a thin uniform plate of which the area is one face. We are also led to speak of the center of gravity of a line or solid. The center of gravity of any geometrical configuration coincides with the center of gravity of a corresponding material body of uniform density.

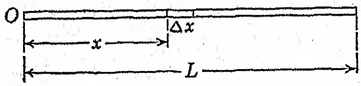

FIG. 92.

Illustration 1. Find the center of gravity of a bar of length L and linear density ρ (mass per unit length) (see Fig. 92). The element of mass is $\rho\,dx$. Then the mean distance of the mass of the bar from O is given by

$$\bar{x} = \frac{\int_0^L \rho x\,dx}{\int_0^L \rho\,dx}.$$

$\bar{x}$ is also the mean moment arm with respect to an axis through O perpendicular to the bar.

If ρ is constant,

$$\bar{x} = \frac{\int_0^L x\,dx}{\int_0^L dx} = \frac{\left.\frac{x^2}{2}\right|_0^L}{\left.x\right|_0^L} = \frac{\frac{1}{2}L^2}{L} = \tfrac{1}{2}L.$$

If the linear density is proportional to the distance from one end, then $\rho = kx$, and

$$\bar{x} = \frac{k\int_0^L x^2\,dx}{k\int_0^L x\,dx} = \frac{\left.\frac{1}{3}x^3\right|_0^L}{\left.\frac{1}{2}x^2\right|_0^L} = \tfrac{2}{3}L.$$

Illustration 2. Find the center of gravity of the area in the first quadrant bounded by the circle $x^2 + y^2 = a^2$.

If we use double integration we have

$$\bar{x} = \frac{\displaystyle\int_0^a \int^{\sqrt{a^2-x^2}} x\,dy\,dx}{\dfrac{\pi a^2}{4}},$$

and

$$\bar{y} = \frac{\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} y\,dy\,dx}{\dfrac{\pi a^2}{4}}.$$

Radicals can be avoided in the evaluation of the numerator of the expression for $\bar{x}$ if the integration is performed first with respect to x and then with respect to y. Thus

$$\bar{x} = \frac{\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} x\,dx\,dy}{\dfrac{\pi a^2}{4}}.$$

The student will evaluate each expression given for $\bar{x}$.

From the symmetry of the figure, $\bar{x} = \bar{y}$, and it is not necessary to evaluate the integral for $\bar{y}$.

In finding the center of gravity in this case, and, indeed, in many cases, it is easier to use single integration than double integration. Thus if we choose, as the element of area, the strip $y\,dx$ parallel to the Y-axis, the moment of this strip about the Y-axis is $xy\,dx$, and

$$\bar{x} = \frac{\displaystyle\int x\,dm}{\displaystyle\int dm} = \frac{\displaystyle\int_0^a xy\,dx}{\displaystyle\int_0^a y\,dx} = \frac{\displaystyle\int_0^a x\sqrt{a^2-x^2}\,dx}{\dfrac{\pi a^2}{4}}.$$

Illustration 3. Find the center of gravity of the solid in the first octant, bounded by the sphere $x^2 + y^2 + z^2 = a^2$.

Using triple integration.

$$\bar{x} = \frac{\displaystyle\int_0^a \int_0^{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2 - y^2}} x\, dz\, dy\, dx}{\dfrac{\pi a^3}{6}}.$$

From considerations of symmetry, $\bar{y} = \bar{z} = \bar{x}$.

Here again it is simpler to use single integration. Choose as element a slab of thickness dx parallel to the YZ-plane. The base of such a slab is a quadrant of a circle of radius $\sqrt{a^2 - x^2}$, where x is the distance of the slab from the YZ-plane. The volume of this elementary slab is

$$\frac{\pi(a^2 - x^2)}{4}\, dx.$$

Hence

$$\bar{x} = \frac{\dfrac{\pi}{4}\displaystyle\int_0^a x(a^2 - x^2)\, dx}{\dfrac{\pi a^3}{6}}.$$

Exercises

Find the coordinates of the center of gravity of:

1. The area between $y = x$ and $y = x^{\frac{3}{2}}$.
2. The smaller of the areas enclosed by $y = 2 - x$ and $x^2 + y^2 = 4$.
3. The area enclosed by $y = 4 - x^2$ and $y = 4 - 2x$.
4. The area enclosed by $y^2 = 4x$ and $y = 3 - x$.
5. The area in the first quadrant bounded by the ellipse, $x = a \cos \theta$, $y = b \sin \theta$. Use single integration.
6. The area in the first quadrant bounded by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
7. The area of a triangle whose vertices are at (0, 0), $(a, 0)$, and (b, c).

8. The volume of the hemisphere generated by revolving about the X-axis the portion of $x^2 + y^2 = r^2$ which lies in the first quadrant. Evidently $\bar{y} = \bar{z} = 0$ and

$$\bar{x} = \frac{\pi \int_0^r xy^2\,dx}{\frac{2}{3}\pi r^3}.$$

9. The volume of a right circular cone whose altitude is 5 and the radius of whose base is 2.

10. The volume of the semi-ellipsoid of revolution generated by revolving one quadrant of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the X-axis.

11. The volume of the paraboloid of revolution generated by revolving about the X-axis the portion of $y^2 = 4x$ between $x = 0$ and $x = 4$.

12. The volume lying in the first octant and included between the cylinders

$$x^2 + y^2 = a^2$$

and

$$x^2 + z^2 = a^2.$$

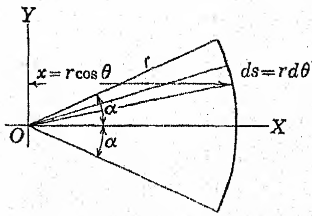

FIG. 93.

13. The volume of one of the wedges cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = x$.

14. A circular arc of radius r and central angle 2α (see Fig. 93).

HINT. The center of gravity lies on the radius which bisects the central angle since this line is an axis of symmetry. Choose this radius as the axis of x and the center of the circle as the origin. Then $\bar{y} = 0$, and

$$\bar{x} = \frac{\int_{-\alpha}^{\alpha} xr\,d\theta}{2r\alpha} = \frac{\int_{-\alpha}^{\alpha} r\cos\theta\, r\,d\theta}{2r\alpha} = \frac{r^2\int_{-\alpha}^{\alpha} \cos\theta\,d\theta}{2r\alpha} = \frac{r\sin\alpha}{\alpha}.$$

This problem can also be solved by using rectangular coordinates. Thus

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \frac{x^2}{y^2}}\, dx = \frac{r}{y}\, dx = \frac{r\, dx}{\sqrt{r^2 - x^2}}$$

$$\bar{x} = \frac{2r \int_{r \cos \alpha}^{r} \frac{x\, dx}{\sqrt{r^2 - x^2}}}{2r\alpha} = \frac{2r^2 \sin \alpha}{2r\alpha} = \frac{r \sin \alpha}{\alpha}.$$

15. The arc of the parabola $y^2 = 4x$, lying in the first quadrant between $x = 0$ and $x = 1$.

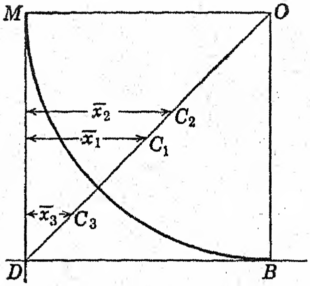

Fig. 94.

16. The arc, lying in the first quadrant, of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

17. The arc of the cycloid

$$x = a(\theta - \sin \theta),\ y = a(1 - \cos \theta)$$

between $x = 0$ and $x = 2\pi a$.

18. The area under one arch of the cycloid,

$$x = a(\theta - \sin \theta),\ y = a(1 - \cos \theta).$$

(Find $\bar{x}$ by inspection.)

19. The lateral surface of a right circular cone whose altitude is h and the radius of whose base is b.

20. The surface of the hemisphere of Exercise 8.

21. The surface of the paraboloid of revolution of Exercise 11.

22.[1] Let OMB, Fig. 94, be a quadrant of a circle of radius r. Let $OMDB$ be a square. Denote by C_1, C_2, and C_3 the centers of gravity of the square, the quadrant of the circle, and the area $MDBM$, respectively; and by A_1, A_2, and A_3 the corresponding areas. Then

$$A_2\bar{x}_2 + A_3\bar{x}_3 = A_1\bar{x}_1.$$

$$\bar{x}_3 = \frac{A_1\bar{x}_1 - A_2\bar{x}_2}{A_3} = 0.223r.$$

$$DC_3 = 0.315\, r.$$

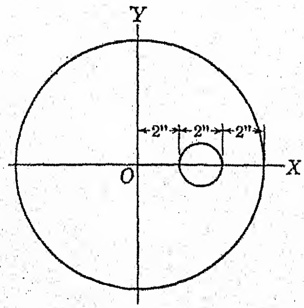

Fig. 95.

23. A circle from which a round hole has been cut (see Fig. 95).

24. A right circular cylinder out of which a circular hole has been drilled parallel to the axis of the cylinder. Figure 96 represents a

[1] Exercise 22 is taken from "Technical Mechanics" by Maurer and Roark.

cross section of the cylinder passing through the axis of the hole and through the axis of the cylinder.

25. Figure 97 represents the meridian section of a shallow metal cup 10 inches in diameter. The thickness of the metal measured from

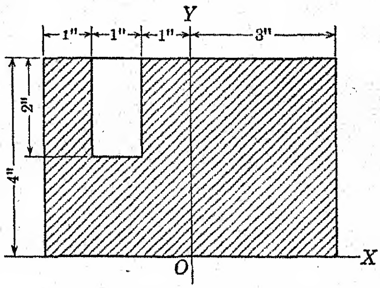

FIG. 96.

the base is given by $1 + \frac{x^2}{10}$, x being measured from the center of the base. Find the height of the center of gravity from the base.

138. Theorems of Pappus. Theorem I. *The area of the surface generated by revolving an arc of a plane curve about an axis in its*

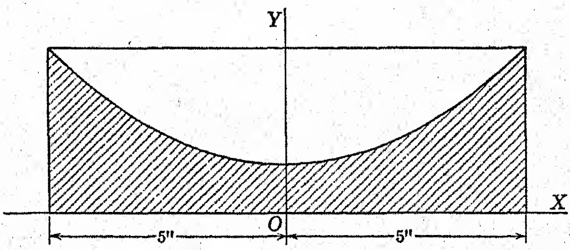

FIG. 97.

plane and not intersecting it is equal to the length of the arc multiplied by the length of the path described by its center of gravity.

Theorem II. *The volume of the solid generated by revolving a plane surface about an axis lying in its plane and not intersecting its boundary is equal to the area of the surface multiplied by the length of the path described by its center of gravity.*

Proof of I. Let ABC, Fig. 98, be an arc of length L lying in the XY-plane. Then $\bar{y}$, the ordinate of its center of gravity, is given by the equation:

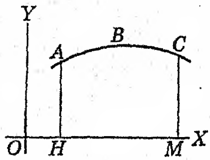

Fig. 98.

$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int y\,ds}{L}. \qquad (1)$$

Then

$$\int y\,ds = \bar{y}L. \qquad (2)$$

The surface generated by revolving the arc ABC about the X-axis is given by

$$S = 2\pi \int y\,ds. \qquad (3)$$

It follows then from (2) and (3) that

$$S = 2\pi\bar{y}L. \qquad (4)$$

But $2\pi\bar{y}$ is the length of the circular path described by the center of gravity of the arc ABC. Hence the theorem is proved.

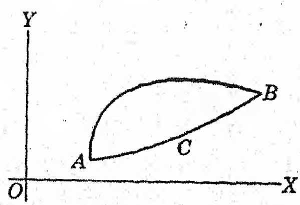

Fig. 99.

Proof of II. Let ABC, Fig. 99, be a plane surface of area A. Then $\bar{y}$, the ordinate of its center of gravity is given by

$$\bar{y} = \frac{\int y\,dA}{A}. \qquad (5)$$

Whence

$$\int y\,dA = A\bar{y}. \qquad (6)$$

Now the volume of the solid generated by the revolution of the area ABC about the X-axis is

$$V = 2\pi \int\int y \, dy \, dx = 2\pi \int y \, dA. \tag{7}$$

It follows from (6) and (7) that

$$V = 2\pi A\bar{y}. \tag{8}$$

Hence the theorem is proved.

Exercises

1. Find the surface of the anchor ring generated by revolving the circle $x^2 + (y - b)^2 = a^2$, $a < b$, about the X-axis.

2. Find the volume of the anchor ring of Exercise 1.

3. Find the volume of a right circular cone, altitude h and radius of base b.

4. Find the lateral surface of a right circular cone, altitude h, and radius of base b.

5. Find the volume of the solid generated by revolving about the X-axis the area of the triangle whose vertices are at $(0, 0)$, $(-a, b)$, and (a, b).

6. Find the volume of the paraboloid generated by revolving about the X-axis the portion of $y^2 = 4x$ between $x = 0$ and $x = 4$.

7. Find the volume of the material in a hollow cylinder whose inner radius is 4 inches, whose outer radius is 6 inches, and whose length is 10 inches.

8. By using a theorem of this section, find the position of the center of gravity of a quadrant of a circular plate whose radius is a.

9. By using a theorem of this section, find the position of the center of gravity of a quadrant of a circular arc whose radius is a.

10. Find the volume of the solid generated by revolving about the X-axis the area bounded by the lines $x = -a$, $x = a$, and the lower half of the circle $x^2 + (y - b)^2 = a^2$, where $b > a$.

139. Center of Gravity. Polar Coordinates. The area of the element $MPQR$, Fig. 87, is $\rho \, d\rho \, d\theta$. Its distance from the Y-axis is $x = \rho \cos \theta$. The product $x \, dm$ is, accordingly, $\rho^2 \cos \theta \, d\rho \, d\theta$ and the mean distance of the area from the Y-axis, or its mean moment arm with respect to the Y-axis, is

$$\bar{x} = \frac{\int\int \rho^2 \cos\theta \, d\rho \, d\theta}{\int\int \rho \, d\rho \, d\theta}.$$

A similar expression can be written for $\bar{y}$, $\rho \cos \theta$ being replaced by $\rho \sin \theta$. If it is advantageous, the integration with respect to θ may be performed first.

Exercises

Find the coordinates of the center of gravity of:

1. The area of one loop of $\rho = a \sin 2\theta$.

2. The area enclosed by $\rho = a(1 + \cos \theta)$.

3. The area of a circular sector of central angle 2α.

4. The area of a portion of a circular ring, Fig. 100, of radii R and r, and of central angle 2α. Denote by C_R the center of gravity of the sector of radius R, by C_r that of the sector of radius r, and by C that of the given portion of the ring. Let the abscissas of these points be x_R, x_r, and $\bar{x}$, respectively, and let the corresponding areas be denoted by A_R, A_r, and A.

HINT. Make use of the result of Exercise 3.

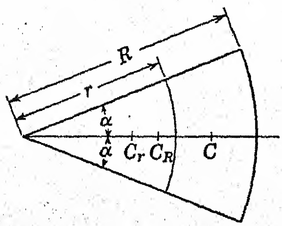

FIG. 100.

5. The area of a segment of a circle of radius r cut off by a chord of length c.

6. The area of the smaller loop of $\rho = 1 + 2 \cos \theta$.

7. The area inside of $\rho = 2 \cos \theta$ and outside of $\rho = \cos \theta$.

8. The smaller area bounded by $\rho = 2a$ and $\rho \cos \theta = a$.

140. Moment of Inertia. Consider a system of n particles of masses $m_1, m_2, \ldots m_n$, respectively, rigidly connected by a framework of negligible mass and rotating about an axis. The system has at a given instant an angular acceleration α. The moment (torque) about the axis of rotation which must be applied to produce this angular acceleration will be computed. The particles $m_1, m_2, \ldots m_n$ move in circles of radii $r_1, r_2, \ldots, r_n$. Let $j_1, j_2, \ldots, j_n$, be, respectively, the tangential accelerations of the particles. The force required to give the mass m_i a tangen-

tial acceleration j_i is m_ij_i. The moment of this force about the axis of rotation is $m_ij_ir_i$. The total moment (torque) required to cause the system to move as described is, accordingly,

$$T = \Sigma m_ij_ir_i.$$

Now $j_i = \alpha r_i$ and $m_ij_ir_i = \alpha m_ir_i^2$. Since α is the same for all the particles, it follows that

$$T = \alpha\Sigma m_ir_i^2.$$

In order to produce unit angular acceleration, a moment equal to $\Sigma m_ir_i^2$ is necessary. The expression $\Sigma m_ir_i^2$ is called the *moment of inertia* of the system of masses with respect to the given axis. It is denoted by the letter I. Thus

$$I = \Sigma m_ir_i^2. \tag{1}$$

The moment of inertia with respect to an axis, of a system of masses, plays the same rôle in the discussion of a motion of rotation as the mass does in the discussion of a motion of translation. In the latter case the force necessary to produce a common linear acceleration j is $j\Sigma m_i$ and that necessary to produce unit linear acceleration is Σm_i; that is, the mass (inertia) of a system is measured by the force necessary to produce *unit linear* acceleration. In like manner *the moment of inertia with respect to an axis, of a mass or a system of masses, is measured by the moment (torque) necessary to produce unit angular acceleration in a motion of rotation about that axis.*

The moment of inertia of a system with respect to an axis evidently depends not only upon the mass but upon the distribution of the mass with respect to the axis. Thus the moment of inertia of a flywheel with most of the mass concentrated in the rim is much greater than the moment of inertia of a uniform solid disk of the same mass and the same radius.

In the case of a continuous mass the sign of summation in (1) is replaced by a sign of integration which may refer to a single, double, or triple integration. Thus

$$\mathrm{I} = \int \mathrm{r}^2\mathrm{dm}. \tag{2}$$

Frequently we shall have occasion to speak of the moment of inertia of a geometrical configuration, such as a line, area, or volume without reference to mass.

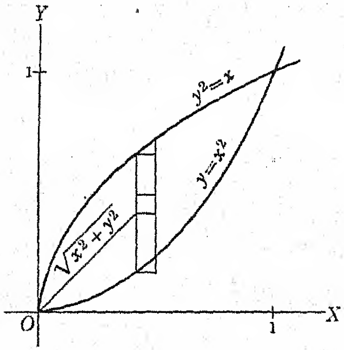

FIG. 101.

Illustration 1. Find the moment of inertia of the uniform plate bounded by the curves $y = x^2$ and $y^2 = x$, Fig. 101, about an axis through the origin perpendicular to the plane of the plate.

Let the mass of the plate per unit area be σ, a constant. The mass of the element $dy\ dx$ is $\sigma\ dy\ dx$. The square of its distance from the axis through O is $x^2 + y^2$. The summation (integration) of $\sigma(x^2 + y^2)dy\ dx$ over the whole plate gives

$$Io = \sigma \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)dy\ dx.$$

If σ is variable, it must be left under the integral sign. If the factor σ is omitted, the expression for the moment of inertia of the mass becomes that for the moment of inertia of the area forming a face of the plate. The moment of inertia of the same plate with respect to the Y-axis is

$$I_y = \sigma \int_0^1 \int_{x^2}^{\sqrt{x}} x^2\ dy\ dx.$$

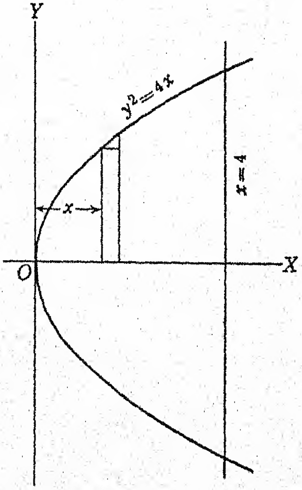

FIG. 102.

Illustration 2. Find the moment of inertia about the Y-axis of the plate of Fig. 102 in the form of a segment of the parabola

$y^2 = 4x$. In this case single integration can be used to advantage, since it is possible to choose a strip whose points are equidistant from the Y-axis. The moment of inertia of the upper half of the plate will be found and multiplied by 2. The element of mass is $\sigma y\,dx$. The square of its distance from the Y-axis is x^2. Hence

$$I_y = 2\sigma\int_0^4 x^2 y\,dx = 4\sigma\int_0^4 x^{\frac{5}{2}}\,dx = \tfrac{8}{7}\sigma x^{\frac{7}{2}}\Big|_0^4 = \frac{1024\sigma}{7}.$$

Illustration 3. Find the moment of inertia of the area of a circle Fig. 103, about an axis through its center and perpendicular to its plane. In this problem it is convenient to choose as element of area a narrow ring of radius r and width Δr. Its area is approximately $2\pi r\Delta r$. Its distance from the center being r, its moment of inertia about the axis in question is approximately $2\pi r^3\Delta r$. The moment of inertia of the whole area is found by summing (integrating) from $r = 0$ to $r = a$.

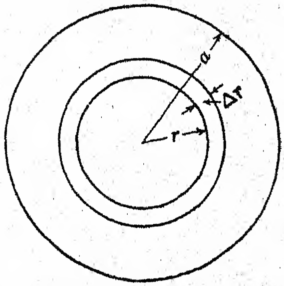

FIG. 103.

$$Io = 2\pi\int_0^a r^3dr = \frac{\pi a^4}{2}.$$

In the case of a uniform circular disk whose mass per unit area is σ, the corresponding moment of inertia is

$$I = \frac{\sigma\pi a^4}{2} = M\frac{a^2}{2},$$

where $M = \sigma\pi a^2$, the mass of the disk.

141. Axis Perpendicular to the Plane of Two Perpendicular Axes. *The sum of the moments of inertia of a plane area (or a plate) in the XY-plane about the X- and Y-axes, respectively, is equal to the moment of inertia of the area (or plate) about an axis through the origin perpendicular to the XY-plane.* For, denoting these respective moments by I_x, I_y, and I_o, we have

$$Io = \int\int(x^2 + y^2)dy\,dx = \int\int x^2\,dy\,dx + \int\int y^2\,dy\,dx,$$

or

$$Io = I_y + I_x.$$

Or, more generally, *the sum of the moments of inertia of a plane area (or a thin plate) about two perpendicular lines in its plane is equal to its moment of inertia about a line perpendicular to these lines at their point of intersection.*

Exercises

Find the moment of inertia of:

1. The area of a rectangle of sides a and b about one of the sides of length a. About an axis through one corner and perpendicular to the plane of the rectangle.

2. The area of a circle about a diameter.
Compare the result with that of *Illustration* 3, §**140.**

3. The area between $y^2 = x$ and $y = x$, about the X-axis. About the Y-axis. About an axis through the origin perpendicular to the XY-plane.

4. The area of a right triangle with legs a and b, about the leg a. About an axis through the vertex of the right angle and perpendicular to the plane of the triangle.

5. The area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, about the X-axis. About the Y-axis. About an axis through the center perpendicular to the XY-plane.

142. Radius of Gyration. The quotient,

$$k^2 = \frac{\int r^2\,dm}{\int dm},$$

of the moment of inertia of a given body with respect to an axis, divided by its mass, is the mean square of the distance of the mass of the body from the axis (see §**134**). If the mass of the body were concentrated at a distance k from the axis, it would have the same moment of inertia with respect to the axis as the given body. k is called the *radius of gyration* of the given body with respect to

the given axis. We also speak of the radius of gyration of a line, area, or volume. It is the root mean square of the distance of the line, area, or volume from the axis in question.

The radius of gyration of the area of a circle, or of a uniform circular disk, with respect to an axis through its center and perpendicular to its plane, is, in accordance with *Illustration* 3, §**140**,

$$k = \frac{a}{\sqrt{2}} = \frac{\sqrt{2}a}{2}$$

143. Compound Pendulum. Let Fig. 104 represent a body free to swing about a horizontal axis through O perpendicular to the plane of the paper. Let $OC = h$ be the line connecting the center of gravity, C, of the body with the point O in the axis and in the same vertical plane with C. Let the variable angle between OC and a vertical line through O be represented by θ. Let M be the mass of the body. The moment about the axis through O, of the force of gravity acting on the body, is then $-Mgh \sin \theta$. The minus sign is used because this moment tends to produce a clockwise rotation when θ is positive. This moment produces the instantaneous angular acceleration α. Another expression for the moment necessary to produce the angular acceleration α is (see §**140**)

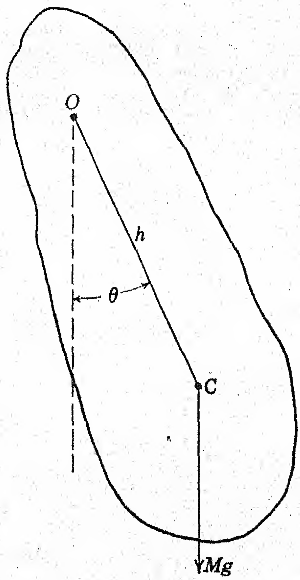

FIG. 104.

$$\alpha \int r^2\, dm = I\alpha = I\frac{d^2\theta}{dt^2},$$

where I is the moment of inertia of the body about the axis of rotation. Hence

$$I\frac{d^2\theta}{dt^2} = -Mgh\sin\theta, \tag{1}$$

or

$$\frac{d^2\theta}{dt^2} = -\frac{M}{I}gh\sin\theta = -\frac{gh}{k^2}\sin\theta. \tag{2}$$

The motion of the compound pendulum is, therefore, the same as that of a simple pendulum of length $\frac{k^2}{h}$ (see §87). If the body swings through a small angle, $\sin\theta$ in (2) can be replaced by θ and the motion is a simple harmonic motion of period

$$\frac{2\pi k}{\sqrt{gh}} = 2\pi\sqrt{\frac{I}{Mgh}}.$$

144. Transfer of Axes. Theorem. *The moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through the center of gravity, increased by the product of the mass by the square of the distance between the axes.*

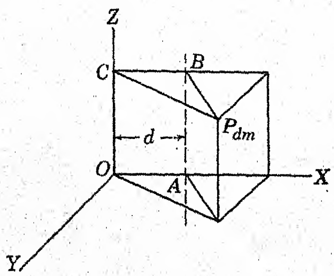

FIG. 105.

Let AB, Fig. 105, be the axis about which the moment of inertia is to be expressed in terms of the moment of inertia about a parallel line through the center of gravity of the body. Choose a system of rectangular axes with the origin O at the center of gravity of the body and with the Z-axis parallel to AB. Take as ZX-plane the plane determined by the Z-axis and the line AB. Consider an element of mass dm at P. The moment of inertia of the body about AB is then

$$I = \int (PB)^2 dm = \int\left[(x-d)^2 + y^2\right] dm,$$

or

$$I = \int (x^2+y^2)dm - 2d\int x\,dm + d^2\int dm. \tag{1}$$

The first term of the right-hand side of (1) is the moment of inertia I_c of the body about the Z-axis, an axis through the center of gravity. The second term, $\int x\,dm$, is the moment of the body with respect to the Y-axis, an axis passing through the center of gravity.

$$\bar{x} = \frac{\int x\,dm}{\int dm}.$$

Since $\bar{x} = 0$, $\int x\,dm = 0$. The last term, $d^2 \int dm$, is d^2M, where M is the mass of the body. Hence

$$\mathbf{I = I_c + Md^2.}$$

Illustration. Find the moment of inertia of a circular plate about an axis through a point in the circumference and perpendicular to the plane of the circle.

The moment of inertia of this area about an axis through the center and perpendicular to the plane of the circle is (see *Illustration* 3, §140)

$$I_c = \frac{\pi a^4}{2}.$$

Hence the moment of inertia of the same area about a parallel axis through a point on the circumference is

$$I = \frac{\pi a^4}{2} + \pi a^2 a^2 = \frac{3\pi a^4}{2}.$$

Exercises

Find the moment of inertia and the radius of gyration of:

1. The area of a rectangle of sides a and b about an axis through the center of gravity and parallel to the sides of length a. About an axis

through the center of gravity and perpendicular to the plane of the rectangle. Use the results of Exercise 1, §**141**.

2. The area of the segment of the parabola $y = x^2$, cut off by $y = 2$, about the Y-axis. About the X-axis. About an axis through the origin perpendicular to the XY-plane.

3. The area of a right triangle with legs a and b about an axis through the center of gravity parallel to the leg a. About an axis through the center of gravity perpendicular to the plane of the triangle. Use the results of Exercise 4, §**141**.

4. The area of a circle about a tangent line. Use the result of Exercise 2, §**141**.

5. The arc of a circle about a diameter.

6. The area of a square of side a, about a line through the center of the square and parallel to one side. About a diagonal. Is it possible to foresee that they are equal?

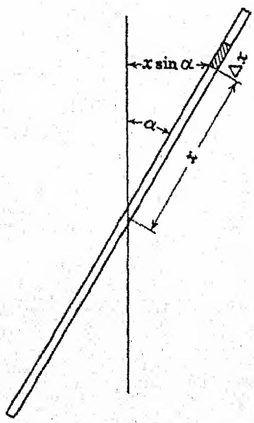

Fig. 106.

7. A uniform bar of length L and linear density ρ about an axis through one end perpendicular to the bar. Find I about a parallel axis through the middle point of the bar.

8. A bar of length L, whose density is proportional to the distance from one end, about an axis perpendicular to the bar through the end of least density.

9. A slender uniform rod, Fig. 106, about a line through its middle point and making an angle α with the rod.

Ans. $I = \frac{1}{12}mL^2 \sin^2 \alpha$, where m is the mass and L is the length of the rod.

Hint. Denote by ρ the linear density. Then

$$I = \rho \int x^2 \sin^2 \alpha \, dx = \rho \sin^2 \alpha \int x^2 \, dx, \text{ with proper limits.}$$

10. The rod of Exercise 9 about a parallel axis through one end.

11. A wire bent into the form of a circular arc, Fig. 107, about the origin. Also find the moments of inertia, I_x and I_y, about the X- and Y-axes, respectively.

$$I = \int_{-\alpha}^{\alpha} r^2\, r\, d\theta;$$

$$I_x = \int_{-\alpha}^{\alpha} r^2 \sin^2\theta\, r\, d\theta;$$

$$I_y = \int_{-\alpha}^{\alpha} r^2 \cos^2\theta\, r\, d\theta.$$

12. A triangle of base b and altitude h about an axis through the vertex parallel to the base. Divide the area into strips parallel to the base and of width dx. The axis of x is drawn from the vertex perpendicular to the base.

$$I = \int_0^h \frac{x^2\, bx\, dx}{h}.$$

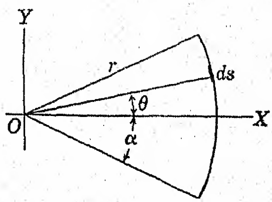

FIG. 107.

13. The lateral surface of a right circular cone about its axis.

14. The surface of a sphere about a diameter.

15. A uniform circular disk 12 inches in diameter from which a circular hole 2 inches in diameter has been cut, the center of the hole being 4 inches from the center of the disk. About a line through the center of the disk perpendicular to its plane. About the diameter through the center of the hole. About a diameter perpendicular to the latter diameter.

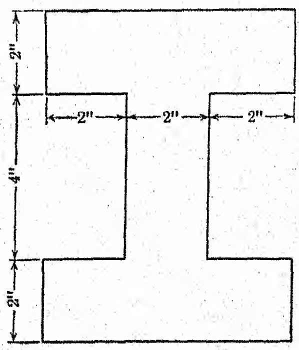

FIG. 108.

16. A uniform circular disk 10 inches in diameter with a square hole of side 2 inches cut out at the center. About a line through the center perpendicular to the plane of the disk. About a diameter of the disk parallel to one side of the square.

17. The cross section of the beam shown in Fig. 108 about a line through the center of gravity of the section and perpendicular to its plane.

18. A uniform rectangular plate of sides a and b swings as a pendulum in a vertical plane about a horizontal axis through O and per-

pendicular to the plane of the rectangle, Fig. 109. Find the period of this pendulum.

19. A uniform elliptical plate with semi-axes of 10 and 5 inches swings as a pendulum about a horizontal axis perpendicular to its plane and passing through a point on the major axis 1 inch from one end. Find the period.

20. A pendulum consists of a thin rectangular strip of metal to which is attached a flat circular disk of the same material and three times as thick. The dimensions are given in Fig. 110. Find the period of the pendulum.

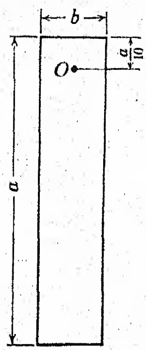

FIG. 109.

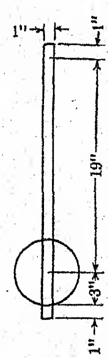

FIG. 110.

145. Moment of Inertia: Polar Coordinates. The moment of inertia of the element of area $r\,dr\,d\theta$ about an axis through the pole perpendicular to the plane of the area is $r^3\,dr\,d\theta$. Hence

$$I = \int\int r^3\,dr\,d\theta.$$

If the moment of inertia of a plate is required, the element of area is to be multiplied by σ, the surface density.

The moments of inertia of a plane area about the X-axis (the polar axis) and the Y-axis are obtained by integrating the product of $r\,dr\,d\theta$ multiplied by $r^2 \sin^2 \theta$ and $r^2 \cos^2 \theta$, respectively.

Exercises

Find the moment of inertia of the following:

1. The area of the cardioid $\rho = a(1 + \cos \theta)$ about an axis through the origin perpendicular to the plane of the cardioid. About the initial line.

2. The area of one loop of $\rho = a \cos 2\theta$ about the initial line.

3. A circular sector of central angle 2α about the radius of symmetry.

4. The arc of the sector of Exercise 3 about the radius of symmetry.

5. The area of Exercise 3 about an axis through the center of the circle and perpendicular to the plane of the sector. About a parallel axis through the center of gravity.

6. The area of one loop of $\rho^2 = a^2 \cos 2\theta$ about an axis through the pole perpendicular to the plane of the area.

146. Moment of Inertia of a Solid. The moment of inertia of a solid of density ρ about the Z-axis is calculated by integrating the product of the element of mass $\rho\, dz\, dy\, dx$ and $x^2 + y^2$, the square of its distance from the Z-axis. Thus

$$I_z = \int\int\int \rho(x^2 + y^2)\, dz\, dy\, dx.$$

Similar expressions can be written for I_x and I_y. If the density is constant ρ can be written before the integral sign.

The moment of inertia of a geometrical solid without reference to its mass is obtained by letting $\rho = 1$. From this result the moment of inertia of the corresponding material solid of uniform density can be obtained by multiplying by the density.

The expressions,

$$I_{yz} = \int\int\int \rho x^2\, dz\, dy\, dx,$$

$$I_{zx} = \int\int\int \rho y^2\, dz\, dy\, dx,$$

$$I_{xy} = \int\int\int \rho z^2\, dz\, dy\, dx,$$

will be called the moments of inertia with respect to the YZ-plane, the XZ-plane, and the XY-plane, respectively. They are the integrals of the product of the element of mass and the square of its distance from the corresponding plane. Frequently, I_{yz}, I_{zx}, and I_{xy} can be found by a single integration by taking as element a plane lamina between two planes parallel to the plane with respect to which the moment is computed. If this is the case, the moment of inertia about the coordinate axes can easily be found by noting that

$$\mathrm{I_z} = \mathrm{I_{yz}} + \mathrm{I_{xz}},$$
$$\mathrm{I_x} = \mathrm{I_{xz}} + \mathrm{I_{xy}},$$
$$\mathrm{I_y} = \mathrm{I_{xy}} + \mathrm{I_{yz}}.$$

That is, *the moment of inertia about the Z-axis is equal to the sum of the moments of inertia with respect to the YZ- and XZ-planes, and so on.*

In general, *the moment of inertia of a body about an axis is equal to the sum of its moments of inertia with respect to two perpendicular planes which intersect in that axis.*

Illustration 1. Find the moment of inertia of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ about each of its axes.

First Method:

$$I_x = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} (y^2 + z^2)\, dz\, dy\, dx.$$

Carry out the integration far enough to see that it is not simple and then note the relative simplicity of the

Second Method: Compute I_{xz}, the moment of inertia with respect to the XZ-plane. Take as element of integration the elliptical plate cut out by the planes $y = y$ and $y = y + \Delta y$. The equation of the intersection of the ellipsoid and the plane $y = y$ is

$$\frac{x^2}{a^2\left(1 - \frac{y^2}{b^2}\right)} + \frac{z^2}{c^2\left(1 - \frac{y^2}{b^2}\right)} = 1.$$

Now the area of an ellipse is π times the product of its semimajor and semiminor axes. Hence the area of this ellipse is

$$\pi a\sqrt{1 - \frac{y^2}{b^2}}\, c\sqrt{1 - \frac{y^2}{b^2}} = \pi ac\left(1 - \frac{y^2}{b^2}\right).$$

The volume of the elliptical plate in question is

$$\pi ac\left(1 - \frac{y^2}{b^2}\right) dy$$

and its moment of inertia with respect to the XZ-plane is

$$\pi a c y^2\left(1 - \frac{y^2}{b^2}\right) dy.$$

The total moment of inertia of the ellipsoid with respect to this plane is then

$$\begin{aligned} I_{xz} &= \pi a c \int_{-b}^{b} y^2\left(1 - \frac{y^2}{b^2}\right) dy \\ &= \pi a c\left(\frac{y^3}{3} - \frac{y^5}{5b^2}\right)\Big|_{-b}^{+b} \\ &= 2\pi a c\left(\frac{b^3}{3} - \frac{b^3}{5}\right) = \frac{4\pi a b^3 c}{15}. \end{aligned}$$

I_{xy} can be written down at once as

$$I_{xy} = \frac{4\pi a b c^3}{15}.$$

Then

$$I_x = I_{xz} + I_{xy} = \frac{4\pi abc}{15}(b^2 + c^2).$$

Interchanging letters,

$$I_y = \frac{4\pi abc}{15}(a^2 + c^2).$$

$$I_z = \frac{4\pi abc}{15}(a^2 + b^2).$$

Some form of the method employed in the following illustration is often useful in avoiding triple integration.

Illustration 2. Find the moment of inertia of the volume of a right circular cone about a line through its vertex and perpendicular to its axis.

Let b and h be the radius of the base and the altitude, respectively. Choose the vertex as origin, and the axis of the cone as axis of x (see Fig. 111). The moment of inertia about the Y-axis is to be found. Consider the circular plate of radius $y = \frac{bx}{h}$ cut out by the

planes $x = x$ and $x = x + \Delta x$. The moment of inertia of this plate about a diameter of its base (choose this diameter parallel to the Y-axis) is equal to $\frac{\pi y^4}{4}\Delta x$, *viz.*, the mass of the plate $\pi y^2 \Delta x$ multiplied by $\frac{y^2}{4}$, the square of its radius of gyration (see Exercise

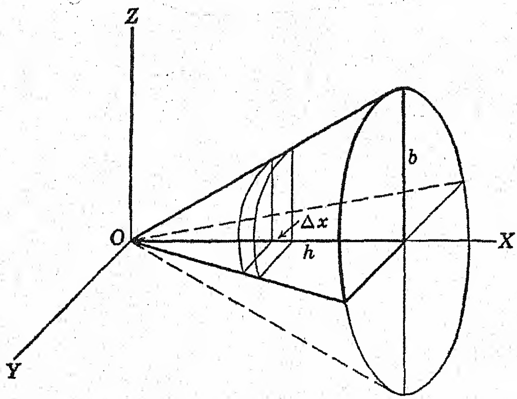

FIG. 111.

2, §**141**) with respect to a diameter. Then the moment of inertia of this plate with respect to the Y-axis is (see §**144**).

$$\frac{\pi y^4}{4}\Delta x + x^2 \pi y^2 \Delta x.$$

The moment of inertia of the entire cone is, therefore,

$$\begin{aligned} I_y &= \int_0^h \left[\frac{\pi y^4}{4} + \pi x^2 y^2\right] dx = \int_0^h \left[\frac{\pi b^4 x^4}{4h^4} + \frac{\pi b^2 x^4}{h^2}\right] dx \\ &= \frac{\pi b^4 h}{20} + \frac{\pi b^2 h^3}{5} = \frac{\pi b^2 h}{20}(b^2 + 4h^2). \end{aligned}$$

The moment of inertia of this cone with respect to its axis can be found by integrating the expression for the moment of inertia of the circular plate between $x = x$ and $x = x + \Delta x$ about an axis

through its center and perpendicular to its plane (see Exercise 1 below).

Exercises

Find the moment of inertia of:

1. A right circular cone about its axis. The radius of the base is b and the altitude is h.

2. A right circular cone about a diameter of the base, using the result of *Illustration* 2.

3. A right circular cylinder, the radius of whose base is r, and whose altitude is h, about a diameter of one base. About a parallel axis through the center of gravity.

4. A right circular cylinder about its axis.

5. A hollow right circular cylinder of outer radius R, inner radius r, and altitude h, about its central axis. About a diameter of one base. About a diameter of the plane section through the center of gravity perpendicular to the axis.

6. A rectangular parallelopiped with edges a, b, and c, about an axis through the center of gravity parallel to one edge.

7. A right rectangular pyramid of base $a \times b$ and of altitude h, about an axis through the center of gravity parallel to the edge a About an axis through the vertex and the center of gravity.

$$\textit{Ans.} \quad I_1 = \frac{abh}{60}(b^2 + \tfrac{3}{4}h^2). \quad I_2 = \frac{abh}{60}(a^2 + b^2).$$

8. A right elliptical cylinder of height L, and having the semimajor and semiminor axes of its cross section equal to a and b, respectively, about an axis through the center of gravity parallel to b.

9. A frustum of a right cone about its axis if the radius of the large base is R, that of the small base is r, and the altitude is h.

$$\textit{Ans.} \quad I = \tfrac{1}{10}\pi h\frac{R^5 - r^5}{R - r}.$$

10. A hollow sphere about a diameter, if the outer radius is R and the inner radius is r.

11. The paraboloid of revolution generated by revolving, about the X-axis, the portion of $y^2 = 4x$ between $x = 0$ and $x = 4$, (*a*) about the X-axis; (*b*) about the Y-axis.

12. A semi-ellipsoid of revolution about a diameter of the base. About the axis of revolution.

13. The anchor ring generated by revolving the circle $(x - a)^2 + y^2 = b^2$, $a > b$, about the Y-axis. Find I_x and I_y.

Write an expression for the error made in assuming the mass of the ring to be concentrated at a distance a from the Y-axis.

14. A circular plate whose meridian section is shown in Fig. 112: (*a*) about the Y-axis; (*b*) about the X-axis.

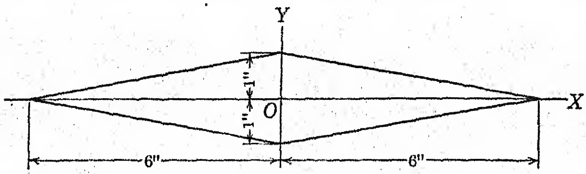

Fig. 112.

15. A pulley is made from a uniform circular disk 8 inches in diameter and 2 inches thick by cutting out a V-shaped groove 1 inch deep and 2 inches wide. Find the moment of inertia and the radius of gyration of the pulley about an axis through the center perpendicular to the plane of the pulley.

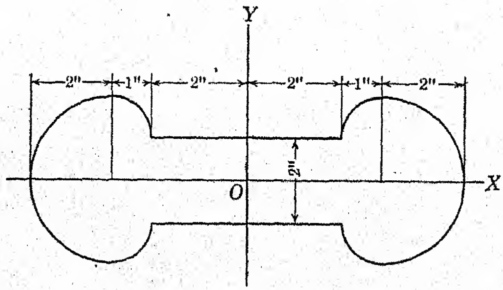

Fig. 113.

16. Find the moment of inertia of the dumb-bell, a meridian section of which is shown in Fig. 113, (*a*) about the X-axis; (*b*) about the Y-axis.

17. The diameter of a right circular cylinder is $2a$ and its altitude is h. Find the moment of inertia of the cylinder about a line parallel to its axis and c units distant from it. $c > a$. Write an expression for the error made in assuming the mass of the cylinder to be concentrated in its axis.

CHAPTER XV

CURVATURE. EVOLUTES. ENVELOPES

147. Curvature. Let PT and QT', Fig. 114, be tangents drawn to the curve APQ at the points P and Q, respectively. Denote the length of the arc PQ by Δs and the angles of inclination of PT and QT' to the positive direction of the X-axis by τ and $\tau + \Delta\tau$, respectively. $\Delta\tau$ gives a rough measure of the deviation from a straight line of that portion of the arc of the curve between the points P and Q. The sharper the bending of the curve between the points P and Q the greater is $\Delta\tau$ for equal values of Δs. The *average curvature* of the curve between the points P and Q is defined by the equation

$$\textit{Average curvature} = \frac{\Delta\tau}{\Delta s}. \qquad (1)$$

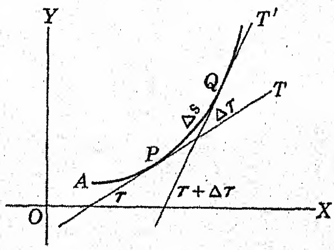

FIG. 114.

The average curvature of a curve between two points P and Q is the average change between these points, per unit length of arc, of the inclination to the X-axis of the tangent line to the curve. Or, more briefly, the average curvature is the average change per unit length of arc, in the inclination of the tangent line.

The curvature at P is defined as the limit of the average curvature between the points Q and P as Q approaches P. On denoting the curvature by K, we have, in accordance with the definition,

$$K = \lim_{\Delta s \to 0} \frac{\Delta\tau}{\Delta s} = \frac{d\tau}{ds}. \qquad (2)$$

The curvature at a point P is then the rate of change at this point of the inclination of the tangent line per unit length of arc. The

curvature is a measure of the amount of bending of a curve in the vicinity of a point.

148. Curvature of a Circle. It is clear that the average curvature of a circle, Fig. 115, is

$$\frac{\Delta\tau}{\Delta s} = \frac{\Delta\tau}{r\,\Delta\tau} = \frac{1}{r}.$$

Hence the average curvature is independent of Δs and consequently the curvature, the limit of the average curvature as Δs approaches zero, is

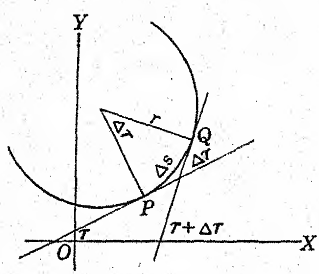

FIG. 115.

$$K = \frac{1}{r}. \tag{1}$$

The curvature of a circle is constant and equal to the reciprocal of its radius.

149. Circle of Curvature. Radius of Curvature. Center of Curvature. Through any point P of a curve infinitely many circles can be drawn which have a common tangent with the curve at P and whose centers are on the concave side of the curve. Of these circles there is one whose curvature is equal to that of the curve at P, *i.e.*, one whose radius is equal to the reciprocal of the curvature at P. This circle is called the *circle of curvature* at the point P. The radius of this circle is called the *radius of curvature*, and its center the *center of curvature*, of the curve at the point P. The radius of curvature is denoted by R and, in accordance with (2), §**147**, its length is

$$\mathbf{R} = \frac{\mathbf{1}}{\mathbf{K}} = \frac{\mathbf{ds}}{\mathbf{d\tau}}. \tag{1}$$

150. Formulas for Curvature and Radius of Curvature: Rectangular Coordinates. For obtaining the curvature at any point on the curve $y = f(x)$, we shall now develop a formula involving the first and second derivatives of y with respect to x. The above formula for curvature K can be written

$$K = \frac{\frac{d\tau}{dx}}{\frac{ds}{dx}} = \frac{\frac{d\tau}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \tag{1}$$

Since

$$\tau = \tan^{-1} \frac{dy}{dx},$$

$$\frac{d\tau}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}.$$

Consequently,

$$\mathbf{K} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}, \tag{2}$$

and by (1), §149,

$$\mathbf{R} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \tag{3}$$

We shall understand by K and R the numerical values of the right-hand members of (2) and (3), respectively, since we shall not be concerned with the algebraic signs of K and R.

Illustration. Find the curvature of $y = x^2$.

$$\frac{dy}{dx} = 2x, \qquad \frac{d^2y}{dx^2} = 2.$$

Substitution in formula (2) gives

$$K = \frac{2}{(1 + 4x^2)^{\frac{3}{2}}}.$$

From this expression it is seen that the maximum curvature

occurs when x is zero, and that the curvature decreases as x increases in numerical value. When $x = 0$, $K = 2$. When $x = \pm 1$, $K = \dfrac{2\sqrt{5}}{25}$.

Exercises

Find the curvature and radius of curvature of each of the curves:

1. $y = 2x - x^2$.
2. $y = x^{\frac{3}{2}}$.
3. $y = \dfrac{1}{x}$.
4. $y = x^2 - x^3$.
5. $y = \dfrac{3}{x^2}$.
6. $y = \sqrt{x}$.
7. $y = 3x^{\frac{1}{3}}$.
8. $y = x^{-\frac{3}{2}}$.
9. $y = \dfrac{1}{\sqrt{x}}$.
10. $(y - 1)^2 = 2p(x + 1)$ at the point $(1, -1)$.
11. $(x + 2)^2 = 2p(y - 1)$ at the point $(-2, 1)$.
12. $y = \log x$ at the point $(1, 0)$.
13. $y = e^x$ at the point $(0, 1)$.
14. If $\rho = f(\theta)$ is the equation of a curve in polar coordinates, show that

$$K = \frac{\rho^2 + 2\left[\dfrac{d\rho}{d\theta}\right]^2 - \rho\dfrac{d^2\rho}{d\theta^2}}{\left[\rho^2 + \left(\dfrac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

Hint.

$$K = \frac{d\tau}{ds} = \frac{\dfrac{d\tau}{d\theta}}{\dfrac{ds}{d\theta}}.$$

$$\tau = \theta + \psi. \qquad \text{(See Fig. 65.)}$$

$$\frac{d\tau}{d\theta} = 1 + \frac{d\psi}{d\theta}.$$

Obtain $\dfrac{d\psi}{d\theta}$ from the relation

$$\psi = \tan^{-1} \frac{\rho}{\dfrac{d\rho}{d\theta}}.$$

$\dfrac{ds}{d\theta}$ is given in §98.

151. Curvature: Parametric Equations. If the equation of a curve is expressed in parametric form, $x = f(t)$, $y = F(t)$, the curvature can be found by differentiating x and y and substituting in (2), §150. t can be eliminated from the result if desired.

Illustration 1. If $x = t$ and $y = t^2$,

$$\frac{dy}{dx} = 2t, \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d(2t)}{dt}\frac{dt}{dx} = 2.$$

Hence

$$K = \frac{2}{(1+4t^2)^{\frac{3}{2}}} = \frac{2}{(1+4x^2)^{\frac{3}{2}}}.$$

Illustration 2. Find the curvature of the ellipse $x = a\cos t$, $y = b\sin t$.

$$\frac{dy}{dx} = -\frac{b}{a}\cot t.$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\frac{d}{dt}\cot t\,\frac{dt}{dx} = \frac{b}{a}\csc^2 t\left[-\frac{1}{a\sin t}\right] = -\frac{b}{a^2}\csc^3 t.$$

$$K = \frac{-\dfrac{b}{a^2}\csc^3 t}{\left[1+\dfrac{b^2}{a^2}\cot^2 t\right]^{\frac{3}{2}}} = \frac{-ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} = \frac{-ab}{\left[\dfrac{a^2y^2}{b^2}+\dfrac{b^2x^2}{a^2}\right]^{\frac{3}{2}}}$$

$$= -\frac{a^4b^4}{(b^4x^2+a^4y^2)^{\frac{3}{2}}}.$$

Exercises

Find the curvature of:

1. $x = \dfrac{t^2+1}{4}$, $y = \dfrac{t^3}{6}$.

2. $x = 3t^2$, $y = 3t - t^3$.

3. $x = \dfrac{4-\sin^2 t}{2}$, $y = 1 + \sin t$.

4. $x = a\cos^3 t$, $y = a\sin^3 t$, at the point corresponding to $t = 15°$.

5. $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

6. $x = a\cosh t$, $y = a\sinh t$.

152. Approximate Formula for Curvature. If $\frac{dy}{dx}$ is small, that is, if the corresponding curve is nearly horizontal, $\left(\frac{dy}{dx}\right)^2$ in formula (2), §150, is very small compared with 1. Hence the

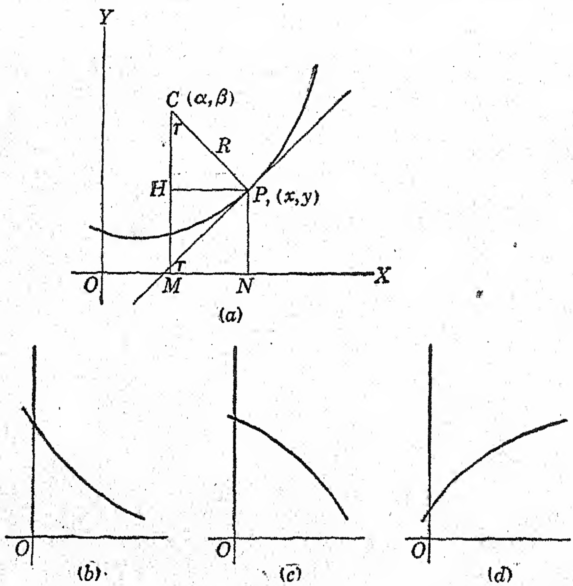

FIG. 116.

denominator differs very little from 1 and the formula for curvature becomes, approximately,

$$K = \frac{d^2y}{dx^2}. \tag{1}$$

This approximate formula for K is used in mechanics in the study of the flexure of beams. The slope of the elastic curve of a beam is so small that $\frac{d^2y}{dx^2}$ can be used for the curvature without appreciable error.

The approximate formula for the radius of curvature R is

$$R = \frac{1}{\dfrac{d^2y}{dx^2}}. \tag{2}$$

153. Center of Curvature. Evolute. Formulas will now be obtained for the coordinates of the center of curvature of a curve corresponding to any point P. Let the coordinates of P be x and y. Denote by α and β the coordinates of the center of curvature of the curve at this point. There are four cases to be considered (see Fig. 116*a*, *b*, *c*, *d*).

In Fig. 116*a*,

$$\alpha = OM = ON - HP = x - R \sin \tau,$$
$$\beta = MC = NP + HC = y + R \cos \tau.$$

Since

$$\tan \tau = \frac{dy}{dx},$$

$$\cos \tau = \frac{1}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}; \quad \sin \tau = \frac{\dfrac{dy}{dx}}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}.$$

Consequently,

$$\alpha = x - \frac{dy}{dx}\,\frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}, \tag{1}$$

and

$$\beta = y + \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}. \tag{2}$$

The student can show that, since $\dfrac{dy}{dx}$ is negative for a descending

curve and positive for an ascending curve, and since $\frac{d^2y}{dx^2}$ is positive when a curve is concave upward and negative when a curve is concave downward, formulas (1) and (2) hold for the three curves represented in Fig. 116*b*, *c*, *d*.

Illustration. Find the coordinates of the center of curvature corresponding to any point on the curve $y = \pm 2\sqrt{x}$. Only the positive sign will be used. If the negative sign is used it will only be necessary to change the sign of β.

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2x^{\frac{3}{2}}}.$$

$$\alpha = x + \frac{1}{\sqrt{x}}\frac{1 + \frac{1}{x}}{\frac{1}{2x^{\frac{3}{2}}}} = 3x + 2.$$

$$\beta = y - \frac{1 + \frac{1}{x}}{\frac{1}{2x^{\frac{3}{2}}}} = y - 2\sqrt{x}(x + 1) = y - y\left[\frac{y^2}{4} + 1\right] = -\frac{y^3}{4}.$$

The equation of the locus of the center of curvature is obtained by eliminating x and y from the equations for α and β and the equation of the original curve. Thus

$$x = \frac{\alpha - 2}{3}; \quad y = -(4\beta)^{\frac{1}{3}}.$$

Substituting in $y^2 = 4x$, we obtain

$$\beta^2 = \tfrac{4}{27}(\alpha - 2)^3,$$

the equation of the locus of the center of curvature. This is the equation of a semicubical parabola whose vertex is at the point (2, 0).

The locus of the center of curvature corresponding to points on a curve is called the evolute of that curve. Its equation is easily

obtained in many cases by eliminating x and y from equations (1) and (2) and the equation of the original curve. Otherwise (1) and (2) constitute its parametric equations, α and β being expressed in terms of the parameters x and y, which are connected by the equation of the original curve.

Exercises

1. Find the evolute of $y = 4x^2$.
2. Find the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

HINT. It will be found that

$$\alpha = \frac{(a^2 - b^2)x^3}{a^4}; \quad \beta = -\frac{(a^2 - b^2)y^3}{b^4}.$$

Whence

$$x = \left[\frac{a^4\alpha}{a^2 - b^2}\right]^{\frac{1}{3}}; \quad y = -\left[\frac{b^4\beta}{a^2 - b^2}\right]^{\frac{1}{3}}.$$

Elimination gives

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

3. Find the parametric equations of the evolute of the cycloid,

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta).$$

Ans. $\alpha = a(\theta + \sin\theta)$, $\beta = -a(1 - \cos\theta)$. Show that the evolute is an equal cycloid with a cusp at the point $(-\pi a, -2a)$.

4. Find the equation of the evolute of

$$x = a(\cos\theta + \theta\sin\theta),$$
$$y = a(\sin\theta - \theta\cos\theta).$$

Ans. $\alpha = a\cos\theta$, $\beta = a\sin\theta$. Discuss.

5. Find the parametric equations of the evolute of $x = \dfrac{t^2 + 1}{4}$, $y = \dfrac{t^3}{6}$.

6. Find the parametric equations of the evolute of $x = \frac{4 - \sin^2 t}{2}$, $y = 1 + \sin t$.

154. Envelopes. If the equation of a curve contains a constant c, infinitely many curves can be obtained by assigning different values to c. Thus

$$(x - c)^2 + y^2 = a^2 \tag{1}$$

is the equation of a circle of radius a whose center is at $(c, 0)$. By assigning different values to c we get a system of equal circles whose centers lie on the X-axis. A constant such as c, to which infinitely many values are assigned, is called a *parameter*. A constant such as a, which is thought of as taking on only one value during the whole discussion, is called an *absolute constant*. We say that equation (1) represents a *family of circles* or a *system of circles* corresponding to the parameter c.

The general equation of a family of curves depending upon a single parameter can be written in the form,

$$f(x, y, c) = 0. \tag{2}$$

Exercises

State the family of curves represented by the following equations containing a parameter:

1. $y = x^2 + c$, c being the parameter.
2. $y = mx + b$, b being the parameter.
3. $y = mx + b$, m being the parameter.
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a being the parameter.
5. $y^2 = m(x + m)$, m being the parameter.
6. $x^2 + y^2 = a^2$, a being the parameter.

Consider again the family of circles (1). Two circles of the family corresponding to the values, c and $c + \Delta c$, of the parameter intersect in the points Q and Q', Fig. 117. We seek the limiting positions of these points of intersection as Δc approaches zero. Clearly, they are the points P and P', respectively, on the lines $y = \pm a$. Such a limiting position of the point of intersection of

two circles of the family is called the point of intersection of two "consecutive" circles of the family. In general, the limiting position of the point of intersection of two curves, $f(x, y, c) = 0$, $f(x, y, c + \Delta c) = 0$, of a family, as Δc approaches zero, is called the point of intersection of "consecutive" curves of the family.

In the case of the family of circles (1) the locus of the points of intersection of "consecutive" circles is the pair of straight lines $y = \pm a$. This locus is called the *envelope* of the family of circles. In general, the *envelope of a family of curves* depending upon one parameter is the locus of the points of intersection of "consecutive" curves of the family. It will be shown in a later chapter that the envelope of a family of curves is tangent to every curve of the family.

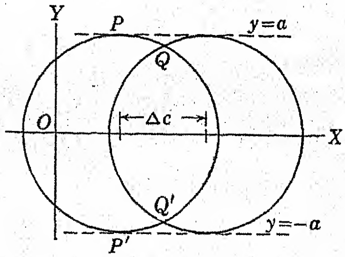

FIG. 117.

Exercise

Draw a number of lines of the family

$$x \cos \alpha + y \sin \alpha = p,$$

where α is the parameter, and sketch the envelope.

A general method of obtaining the envelope of a family of curves will now be given.

The equation of a curve of the family is

$$f(x, y, c) = 0, \tag{3}$$

where c has any fixed value. The envelope is the locus of the limiting position of the point of intersection of any curve (3) of the family with a neighboring curve, such as

$$f(x, y, c + \Delta c) = 0, \tag{4}$$

as the second curve is made to approach the first by letting Δc approach zero. The coordinates of the points of intersection of the curves representing equations (3) and (4) satisfy

$$f(x, y, c + \Delta c) - f(x, y, c) = 0. \tag{5}$$

Then they satisfy

$$\frac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c} = 0, \tag{6}$$

since Δc does not depend on either x or y. Then the coordinates of the limiting positions of these points of intersection satisfy

$$\lim_{\Delta c \to 0} \frac{f(x, y, c + \Delta c) - f(x, y, c)}{\Delta c} = 0.$$

The first member of this equation is the derivative of $f(x, y, c)$ with respect to c. It may be written in the form,

$$\frac{\partial f(x, y, c)}{\partial c} = 0. \tag{7}$$

The differentiation is performed with respect to c, x and y being treated as constants. The point of intersection also lies on (3). Hence the equation of its locus is obtained by eliminating c between (3) and (7).

Illustration 1. Find the equation of the envelope of the family of circles $(x - c)^2 + y^2 = a^2$, c being the parameter.

The equation of the curve written in the form $f(x, y, c) = 0$ is

$$(x - c)^2 + y^2 - a^2 = 0. \tag{I}$$

Differentiating with respect to c,

$$-2(x - c) = 0. \tag{II}$$

The elimination of c between (I) and (II) gives

$$y^2 = a^2,$$

or

$$y = \pm a,$$

as the envelope.

Illustration 2. Find the equation of the envelope of the family of lines, $x \cos \alpha + y \sin \alpha = p$, α being the parameter.

On differentiating the first member of

$$x \cos \alpha + y \sin \alpha - p = 0 \tag{I}$$

with respect to α we obtain

$$-x \sin \alpha + y \cos \alpha = 0. \qquad \text{(II)}$$

The result of eliminating α between (I) and (II) is

$$x^2 + y^2 = p^2,$$

a circle of radius p about the origin as center.

Exercises

1. Find the envelope of the family of straight lines $y = mx + \frac{p}{m}$, where m is the parameter. Draw figure.

2. Find the envelope of the family of lines $y = x \tan \alpha + a \sec \alpha$, where α is the parameter. Draw figure.

3. Find the envelope of the family of parabolas $y^2 = c(x - c)$, c being the parameter.

4. Find the envelope of the family of lines of constant length whose extremities lie in two perpendicular lines.

5. Find the envelope of $y = px - p^2$, p being the parameter. Draw figure.

6. Find the envelope of the family of curves $(x - c)^2 + y^2 = 4pc$, c being the parameter. Draw figure.

7. The equation of the path of a projectile fired with an initial velocity v_0 which makes an angle α with the horizontal is

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}.$$

Find the envelope of the family of paths obtained by considering α a parameter. *Ans.* $y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$.

8. The equation of the normal to $y^2 = 4x$ at the point P, whose coordinates are x_1 and y_1, is

$$y - y_1 = -\frac{y_1}{2}(x - x_1).$$

Since $y_1^2 = 4x_1$, this may be written

$$y_1 x + 2y - \frac{y_1^3}{4} - 2y_1 = 0.$$

Find the equation of the envelope of the normals as P moves along the curve.

HINT. On differentiating with respect to the parameter y_1 we obtain

$$y_1 = \pm 2\frac{\sqrt{x-2}}{\sqrt{3}}.$$

On substituting this value of y_1 in the equation of the normal and squaring we obtain

$$y^2 = \frac{4(x-2)^3}{27}.$$

This is the evolute of the parabola as we have seen in **§153.**

9. Find the equation of the envelope of

$$(x-t)^2 + (y+t)^2 = t^2 + 2.$$

10. Find the equation of the envelope of $(x-t)^2 + y^2 = 1 - t^2$.

155. The Evolute as the Envelope of the Normals. In Exercise 8 of **§154** it was seen that the evolute of a parabola is the envelope of its normals. This is true for any curve. The result is fairly evident from an examination of the curves of the exercises of **§153** and their evolutes. It will be shown that the normals to a curve are tangent to its evolute.

The parametric equations of the evolute are

$$\alpha = x - R\sin\tau, \tag{1}$$
$$\beta = y + R\cos\tau. \tag{2}$$

On differentiating with respect to the variable s, which is permissible, since x, y, R, and τ are all functions of s, we obtain

$$\frac{d\alpha}{ds} = \frac{dx}{ds} - \frac{dR}{ds}\sin\tau - R\cos\tau\,\frac{d\tau}{ds},$$
$$\frac{d\beta}{ds} = \frac{dy}{ds} + \frac{dR}{ds}\cos\tau - R\sin\tau\,\frac{d\tau}{ds}.$$

Now

$$\frac{dx}{ds} = \cos\tau,$$

$$\frac{dy}{ds} = \sin\tau,$$

$$\frac{d\tau}{ds} = \frac{1}{R}.$$

Then the foregoing equations become

$$\frac{d\alpha}{ds} = -\frac{dR}{ds}\sin\tau,$$

$$\frac{d\beta}{ds} = \frac{dR}{ds}\cos\tau.$$

Hence the slope of the tangent to the evolute is

$$\frac{d\beta}{d\alpha} = -\cot\tau. \tag{3}$$

Therefore the tangent to the evolute is parallel to the normal to

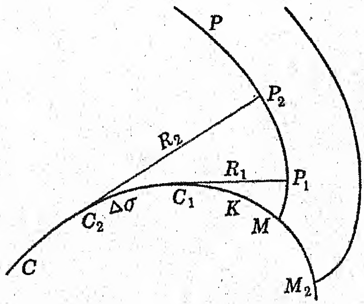

Fig. 118.

the curve at the point (x, y) to which (α, β) corresponds. But the normal to the curve at (x, y) passes through (α, β). Hence it is tangent to the evolute at this point.

It can also be shown that if C_1 and C_2, Fig. 118, are the centers of curvature corresponding to the points P_1 and P_2, the length of the arc C_1C_2 of the evolute is equal to the difference in the lengths

of the radii of curvature, R_1 and R_2, for from the above values of $d\alpha$ and $d\beta$ it follows that

$$\sqrt{d\alpha^2 + d\beta^2} = dR.$$

But $\sqrt{d\alpha^2 + d\beta^2}$ is the differential of the arc of the evolute. Call it $d\sigma$. Then $d\sigma = dR$, and hence on integrating $\sigma = R + C$. σ and R are functions of s, the arc of the given curve. Then corresponding to a change $\Delta s(=$ arc $P_1P_2)$ in s, σ and R will take on the increments $\Delta\sigma$ and ΔR which are equal by the foregoing equation. But $\Delta\sigma =$ arc C_1C_2, and $\Delta R = R_2 - R_1$. Hence arc C_1C_2 equals $R_2 - R_1$.

156. Involutes. In Fig. 118, suppose that one end of a string is fastened at C and that it is stretched along the curve CC_2C_1KM. If now the string be unwound, always being kept taut, the point M will, in accordance with the properties of the evolute, trace out the curve MP_1P_2P. This curve is called the *involute* of the curve KC_1C_2C. If longer or shorter lengths of string such as CKM_2 be used, other involutes will be traced. In fact, to a given curve there correspond infinitely many involutes. The given curve is the evolute of each of these involutes. We see that while a given curve has but one evolute it has infinitely many involutes.

In Exercise 4, §**153**, the circle $x = a\cos\theta$, $y = a\sin\theta$ was found as the evolute of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$. Then the latter curve is an involute of the circle. The student will draw a figure showing a position of the string as it would be unwound to generate the involute and indicate the angle θ.

CHAPTER XVI

SERIES. TAYLOR'S AND MACLAURIN'S THEOREMS. INDETERMINATE FORMS

157. Infinite Series. An expression of the form

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \tag{1}$$

is called an infinite series. The terms $u_1, u_2, \ldots, u_n, \ldots$ of an infinite series are unlimited in number. They may be either constants or functions of one or more variables. The following are examples of infinite series:

$$1 + x + x^2 + x^3 + \cdots$$

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Let s_n denote the sum of the first n terms of the series (1). Thus

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n. \tag{2}$$

If s_n approaches a limit s as n increases without limit, this limit is called the sum of the series and the series is said to be *convergent*. If s_n does not approach a limit as n becomes infinite, the series is said to be *divergent*. If a series is convergent, the sum of a few terms will frequently be a satisfactory approximation to s. If a series is divergent, it is not in general suitable for purposes of calculation. In more advanced courses it is shown how use can be made of certain divergent series.

An elementary type of infinite series is one whose terms form a geometrical progression. For example,

$$a + ar + ar^2 + \cdots + ar^n + \cdots. \tag{3}$$

For this series

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} \tag{4}$$

$$= a\frac{1 - r^n}{1 - r} = \frac{a}{1 - r} - a\frac{r^n}{1 - r}. \tag{5}$$

If[1] $|r| < 1$ it is clear from (5) that

$$\lim_{n \to \infty} s_n = \frac{a}{1 - r}.$$

Hence the series (3) is convergent if the numerical value of r is less than one. If $|r| > 1$ it is clear from (5) that s_n does not approach a limit as n becomes infinite. Hence, the series (3) is divergent if the numerical value of r is greater than 1.

If $r = 1$, $s_n = na$ and s_n does not approach a limit as n becomes infinite.

If $r = -1$, s_n is equal to a if n is odd and is equal to 0 if n is even. Thus s_n continually oscillates between the two values a and 0 as n increases and does not approach a limit.

We conclude that the series (3) is convergent if $|r| < 1$ and that it is divergent if $|r| \geqq 1$.

158. Convergence of Series with Positive Terms. Theorem 1. *If s_n, the sum of the first n terms of the series of positive terms*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

continually increases as n increases but always remains less than some fixed number M, then s_n approaches a limit L less than or equal to M as n becomes infinite. The series is therefore convergent.

In more advanced courses a proof is given of this theorem with the aid of a certain axiom. The theorem will be accepted here as a working principle whose significance is shown in Fig. 119 and whose truth appears to be obvious.

[1] The symbol $|r|$ denotes the numerical or absolute value of r. Thus $|-2| = 2$ and $|2| = 2$.

159. Comparison Test. Theorem 2. *If each term of a series of positive terms is less than or equal to the corresponding term of a known convergent series of positive terms the series is convergent.*

Let

$$U_1 + U_2 + \cdots + U_n + \cdots \tag{1}$$

be a series of positive terms that is known to be convergent, and let each term of the series of positive terms

$$u_1 + u_2 + \cdots + u_n + \cdots \tag{2}$$

be less than or equal to the corresponding term of the series (1),

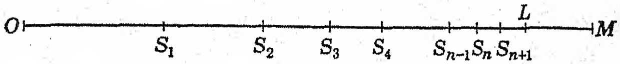

FIG. 119.

that is, $u_1 \leqq U_1$, $u_2 \leqq U_2$, and so on. It is to be proved that the series (2) is convergent.

Let

$$S_n = U_1 + U_2 + \cdots + U_n, \tag{3}$$
$$s_n = u_1 + u_2 + \cdots + u_n. \tag{4}$$

Since the series (1) is convergent, the sum S_n has a limit as n becomes infinite. Let this limit be called S. Since each term entering into the sum s_n is less than or equal to the corresponding term entering into the sum S_n, it follows that

$$s_n \leqq S_n.$$

But

$$S_n < S.$$

Therefore $s_n < S$ no matter how large n is taken. But s_n is continually increasing with n. Then in accordance with Theorem 1, s_n has a limit s as n becomes infinite and this limit is less than or at most equal to S.

Theorem 3. *If each term of a series of positive terms is greater than or equal to the corresponding term of a known divergent series of positive terms, the series is divergent.*

Use the notation employed in the proof of Theorem 2. Here $u_1 \geqq U_1$, $u_2 \geqq U_2$, $\cdots$ and $\lim\limits_{n\to\infty} S_n$ does not exist. In fact, S_n increases beyond all limit. Otherwise the series $U_1 + U_2 + \cdots + U_n + \cdots$ would converge in accordance with Theorem 1. Now $s_n \geqq S_n$ no matter how large n is taken. Hence s_n increases without limit as n increases. Hence the series $u_1 + u_2 + \cdots + u_n + \cdots$ is divergent.

160. A Necessary Condition for Convergence. Theorem 4. *If the series*

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is convergent, $\lim\limits_{n\to\infty} u_n = 0.$

The convergence of s_n to a limit s with increasing n implies that u_n must become arbitrarily small as n increases. Otherwise the difference between s_n and s would not become and remain arbitrarily small.

If $\lim\limits_{n\to\infty} u_n$ is not zero, it follows that the series is divergent.

However, if $\lim\limits_{n\to\infty} u_n = 0$, it does not follow that the series is convergent. The condition is merely a necessary condition for convergence but not a sufficient one as the following example will show. The series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \cdots + \frac{1}{n} + \cdots \tag{1}$$

is divergent, although

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

That the series is divergent can easily be seen by comparing it with the series

$$\begin{aligned} 1 &+ \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} \\ &+ \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \cdots \end{aligned} \tag{2}$$

and noting that each term of the series (1) is greater than or equal to the corresponding term of the series (2). If $n > 2$ the sum of

the first n terms of (1) is, therefore, greater than the sum of the first n terms of (2). The sum of the third and fourth terms of (2) is $\frac{1}{2}$, that of the next four is $\frac{1}{2}$, that of the next eight is $\frac{1}{2}$, and so on. It is evident that s_n, formed for the series (2), becomes infinite as n increases without limit. Much more then is the series (1) divergent.

161. Alternating Series. Theorem 5. *If the terms of a series are alternately positive and negative, if each term is less than the preceding in numerical value, and if* $\lim\limits_{n \to \infty} u_n = 0$, *the series is convergent.*

A series of this type is the following:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots.$$

The reason for the convergence of such an alternating series can be seen as follows. Denote by s_n the sum of the first n terms and

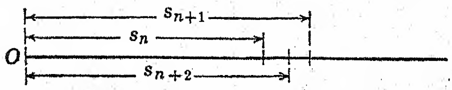

Fig. 120.

suppose the $(n+1)$th term positive (see Fig. 120). Then, since the terms are constantly decreasing,

$$s_{n+1} > s_n; \quad s_{n+2} < s_{n+1}; \quad s_{n+2} > s_n.$$

It is clear that as n increases s_n oscillates back and forth, but always within narrower and narrower limits, since the terms are constantly decreasing. As n becomes infinite, the amount of this oscillation approaches zero, since

$$\lim_{n \to \infty} u_n = 0.$$

Therefore s_n approaches a limit.

The error made in taking the sum of the first n terms of such a series, as the sum of the series, is less in numerical value than the first term neglected. This follows from the fact that the limit s always lies between s_n and s_{n+1} in such an alternating series, no matter what n may be (see Fig. 120).

162. The Ratio Test. **Theorem 6.** *The series* $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ *is convergent if*

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$$

and divergent if

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1.$$

If $\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = 1$ *the series may be convergent or it may be divergent.*

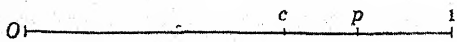

FIG. 121.

The terms of the series will at first be assumed to be positive. Let

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = c < 1.$$

Choose a number p between c and 1 (see Fig. 121). We can find a number m such that $\frac{u_{n+1}}{u_n} < p$ for all values of n greater than or equal to m. Hence

$$\frac{u_{m+1}}{u_m} < p,$$

or

$$u_{m+1} < u_m p.$$

Also

$$\begin{aligned} u_{m+2} &< u_{m+1}p < u_m p^2 \\ u_{m+3} &< u_{m+2}p < u_m p^3 \\ &\cdots\cdots \\ &\cdots\cdots \\ u_{m+q} &< u_{m+q-1}p < u_m p^q. \end{aligned}$$

Hence the terms of the series from u_{m+1} on, are less than the corresponding terms of the series

$$u_m p + u_m p^2 + u_m p^3 + \cdots.$$

The latter series is convergent since it is an infinite geometrical progression whose ratio is less than 1. Hence by the comparison test the series

$$u_{m+1} + u_{m+2} + u_{m+3} + \cdots$$

is convergent. Thus the series formed by neglecting the first m terms of the given series is convergent. Hence the given series is convergent.

If the terms of the given series are not all positive, the test is to be applied to the series

$$|u_1| + |u_2| + \cdots + |u_n| + \cdots \tag{1}$$

whose terms are the absolute values of those of the given series. It can be shown (see **§163**) that the given series is convergent if the series (1) is convergent.

Let

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = b > 1,$$

where the terms of the series may be positive or negative. Choose a number r between 1 and b. It is possible to find a number m such that $\left|\frac{u_{n+1}}{u_n}\right| > r$ if $n \geqq m$. Then

$$|u_{m+1}| > |u_m|r,\ |u_{m+2}| > |u_m|r^2,\ |u_{m+3}| > |u_m|r^3, \cdots.$$

Since $r > 1$ it is clear that $\lim\limits_{n\to\infty} u_n$ does not approach zero. Hence the series is divergent.

If $\lim\limits_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = 1$, the series may be convergent or it may be divergent. Consider the divergent series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots.$$

For this series

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\dfrac{1}{n+1}}{\dfrac{1}{n}}=\lim_{n\to\infty}\frac{n}{n+1}=1.$$

The series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

will be shown to be convergent in *Illustration* 2, §164. For this series

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\dfrac{1}{(n+1)^2}}{\dfrac{1}{n^2}}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2=1.$$

Since one of the series just considered diverges and the other converges, it follows that the ratio test fails when $\lim\limits_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right|=1.$

163. Series of Positive and Negative Terms. Theorem 7. *If the terms of the series*

$$u_1+u_2+u_3+\cdots \tag{1}$$

are part positive and part negative and if the series of the absolute values of the terms

$$|u_1|+|u_2|+|u_3|+\cdots \tag{2}$$

is convergent, then the series (1) *is convergent.*

Select the positive terms from (1), denoting them by p_1, p_2, p_3, etc. and form the series

$$p_1+p_2+p_3+\cdots. \tag{3}$$

Select the negative terms from (1) denoting them by $-q_1$, $-q_2$, $-q_3$, etc., q_1, q_2, q_3, etc. being positive numbers. Form the series

$$q_1+q_2+q_3+\cdots. \tag{4}$$

Let

$$s_n = u_1 + u_2 + \cdots + u_n.$$

In s_n there will be included, let us say, m terms of series (3) and r terms of series (4). Let

$$S_m = p_1 + p_2 + \cdots + p_m$$
$$T_r = q_1 + q_2 + \cdots + q_r.$$

Then

$$s_n = S_m - T_r.$$

Let

$$\Sigma_n = |u_1| + |u_2| + |u_3| + \cdots + |u_n|$$

and

$$\lim_{n\to\infty} \Sigma_n = L,$$

the series (2) being convergent by hypothesis. Now

$$S_m < \Sigma_n < L, \qquad T_r < \Sigma_n < L.$$

Hence by the theorem of **§158** each of the quantities S_m and T_r approaches a limit. Hence $\lim s_n = \lim S_m - \lim T_r$ exists since each of the limits on the right exists. The series (1) is therefore convergent.

164. Applications of the Tests for Convergence. There is no one test for convergence which can be applied with certainty of success to any series whatever. The tests which have been given can frequently be successfully applied. There are many other tests that will be found in more extensive treatments of infinite series. In determining whether or not a given series is convergent it is suggested that the following procedure be observed in general:

1. Determine whether or not $\lim_{n\to\infty} u_n = 0$. If not, the series is divergent.

2. Determine whether or not the series is an alternating series as described in the theorem of **§161**.

3. If the series is not an alternating series, try the ratio test. This will fail if the limit of the ratio in question is 1.

4. If the ratio test fails, try a comparison test.

Illustration 1. Test the series

$$1 - \tfrac{2}{1} + \tfrac{3}{2} - \tfrac{4}{3} + \tfrac{5}{4} - \tfrac{6}{5} + \cdots$$

for convergence.

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n}{n-1} = 1.$$

Hence the series is divergent. (See **§160.**)

Illustration 2. Test the series

$$1 + \frac{1}{2^t} + \frac{1}{3^t} + \frac{1}{4^t} + \cdots \qquad (a)$$

for convergence.

If $t > 0$, $\lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} \dfrac{1}{n^t} = 0$. The ratio test will be found to fail. It is consequently necessary to make use of comparison tests.

If $t = 1$, we have seen, **§160,** that this series is divergent. If $t < 1$, each term of (a) is greater than the corresponding term of (1), **§160,** and hence (a) is divergent. If $t > 1$ we can compare (a) with

$$1 + \frac{1}{2^t} + \frac{1}{2^t} + \frac{1}{4^t} + \frac{1}{4^t} + \frac{1}{4^t} + \frac{1}{4^t} + \cdots \qquad (b)$$

Each term of (a) is less than or equal to the corresponding term of (b). But (b) is convergent since it can be written

$$1 + 2\left(\frac{1}{2^t}\right) + 4\left(\frac{1}{4^t}\right) + 8\left(\frac{1}{8^t}\right) + \cdots$$

or

$$1 + \frac{2}{2^t} + \frac{4}{4^t} + \frac{8}{8^t} + \frac{16}{16^t} + \cdots$$

which is a geometric series whose ratio $\frac{2}{2^t}$ is less than 1. Hence (*a*) is convergent when $t > 1$. It has been shown that:

(*a*) is divergent if $t \leqq 1$.
(*a*) is convergent if $t > 1$.

The convergence or divergence of other series can often be established by comparing them with the series (*a*).

Illustration 3. Test the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

for convergence.

$$u_n = \frac{1}{n!}.$$

Hence

$$\lim_{n \to \infty} \left| \frac{u_n + 1}{u_n} \right| = \lim_{n \to \infty} \frac{\dfrac{1}{(n+1)!}}{\dfrac{1}{n!}} = \lim_{n \to \infty} \frac{1}{n+1} = 0.$$

The series is therefore convergent (Theorem 6).

If the sum of the first $(n - 1)$ terms is taken as the sum of the series, the error is less than

$$\frac{1}{n!}\left(1 + \frac{1}{n} + \frac{1}{n^2} + \cdots \right) = \frac{1}{n!}\left(\frac{1}{1 - \frac{1}{n}}\right).$$

Illustration 4. For what values of x, if any, is the series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

convergent?

$$|u_n| = \left| \frac{x^{2n-1}}{(2n-1)!} \right|.$$

Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{x^{2n+1}}{(2n+1)!}}{\dfrac{x^{2n-1}}{(2n-1)!}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{2n(2n+1)}\right| = 0$$

for all finite values of x. Hence the series is convergent of all finite values of x, positive, or negative (Theorem 6).

The same conclusion may be drawn by applying Theorem 5.

Illustration 5. For what values of x is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$$

convergent?

$$|u_n| = \left|\frac{x^n}{n}\right|.$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{x^{n+1}}{n+1}}{\dfrac{x^n}{n}}\right| = \lim_{n\to\infty}\left|x\frac{n}{n+1}\right| = |x|.$$

The series is therefore convergent if $|x| < 1$. Furthermore, it is convergent if $x = 1$ (Theorem 5), and divergent if $x = -1$. See series (1), §160.

Exercises

Test the following series for convergence and find the sum, correct to three decimal places, of those series which converge sufficiently rapidly.

1. $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \cdots.$
2. $\frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \frac{5}{11} - \cdots.$
3. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots.$
4. $\dfrac{2!}{10} + \dfrac{3!}{10^2} + \dfrac{4!}{10^3} + \dfrac{5!}{10^4} + \cdots.$
5. $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \dfrac{4}{5!} + \cdots.$
6. $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \cdots.$

7. $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots.$

8. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots.$

9. $\frac{1}{1\cdot2} + \frac{1}{3\cdot4} + \frac{1}{5\cdot6} + \cdots.$

10. $\frac{1}{2} - \frac{1}{4\cdot2} + \frac{1}{8\cdot3} - \frac{1}{16\cdot4} + \frac{1}{32\cdot5} - \cdots.$

11. $\frac{1}{4} + \frac{1}{16\cdot2} + \frac{1}{64\cdot3} + \cdots.$

12. $1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \frac{1}{9!} + \cdots.$

13. $\frac{1}{2} - \frac{1}{3(2^3)} + \frac{1}{5(2^5)} - \frac{1}{7(2^7)} + \cdots.$

For what values of x are the following series convergent? Find the sum of each series correct to three decimal places if $x = 0.2$.

14. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$

15. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$

16. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$

17. $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots.$

18. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$

165. Power Series. An infinite series of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

where the coefficients $a_0, a_1, a_2, \ldots, a_n, \ldots$ are constants, is called a power series in x. One of the form

$$a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots$$

is called a power series in $(x - a)$.

Formulas[1] will now be derived for obtaining the expansion of a given function in a power series in x or in $(x - a)$.

[1] The derivation of these formulas, given in §§166 to 169, may be replaced by the briefer but less rigorous derivation given in §170.

166. Rolle's Theorem. *If $f(x)$ is a single-valued function of x in the interval*[1] *(a, b), if it has a derivative $f'(x)$ at every interior point of this interval, and if $f(a) = 0$ and $f(b) = 0$, then $f(x)$ is equal to zero for at least one value of x lying between a and b.*

The graph of the function $f(x)$ either crosses or touches the X-axis at $x = a$ and $x = b$, since by hypothesis $f(a) = 0$ and $f(b) = 0$. Unless $f(x)$ remains always equal to zero in the interval (a, b) the graph of the function $f(x)$ must have at least one maximum or minimum point between $x = a$ and $x = b$. Two possible situations are shown in Figs. 122 and 123. As $f(x)$ has a derivative at every point between $x = a$ and $x = b$ the slope of the curve must be zero at such a maximum or minimum point. That is, $f'(x_1) = 0$ for at least one value x_1 of x satisfying the condition $a < x_1 < b$.

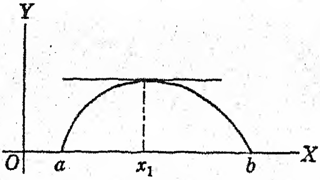

FIG. 122.

The fact that $f(x)$ is assumed to be continuous excludes the possibility of situations such as those illustrated in Figs. 124 and 125 while the fact that $f(x)$ is assumed to have a derivative excludes functions such as those whose graphs are shown in Figs. 126 and 127.

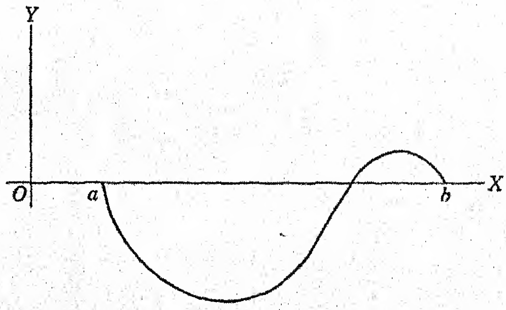

FIG. 123.

In the proof of this theorem geometrical reasoning has been used. A proof based upon analytical reasoning and not relying upon

[1] A variable x is said to lie in the interval (a, b) if $a \leqq x \leqq b$.

geometrical intuition can be given. Such a proof is better suited to a more advanced course and is omitted here.

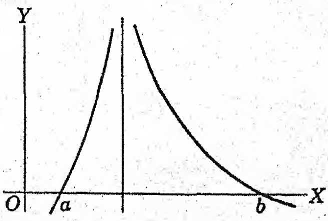

FIG. 124.

167. Law of the Mean. Let $f(x)$ be a continuous single-valued function of x in the interval (a, b) and let it have a derivative at every interior point of this interval. It is then apparent from Fig. 128 that the tangent line to the curve AB at some point P between A and B will be parallel to the secant line AB.

Theorem. *If $f(x)$ is single-valued and continuous in the interval*

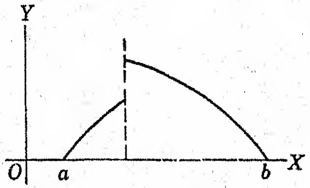

FIG. 125.

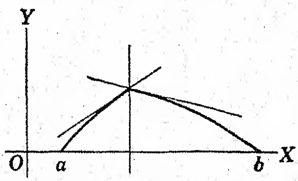

FIG. 126

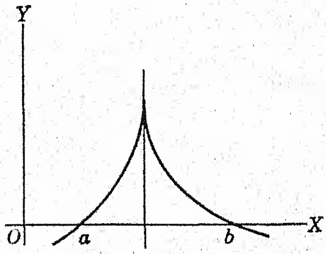

FIG. 127.

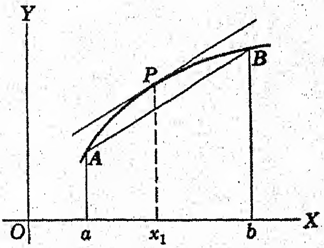

FIG. 128.

(a, b) and has a derivative at every interior point, then

$$f'(x_1) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

where $a < x_1 < b$.

Equation (1) can also be written in the form

$$f(b) = f(a) + (b - a)f'(x_1). \tag{2}$$

It is easy to give an analytical proof of this theorem whose truth seems apparent from the geometrical reasoning used above. Let a number S be defined by the equation

$$f(b) = f(a) + (b - a)S. \tag{3}$$

It will be proved that S is equal to $f'(x_1)$, $a < x_1 < b$. Write (3) in the form

$$f(b) - f(a) - (b - a)S = 0 \tag{4}$$

and introduce a new function $\phi(x)$ which is defined by the following equation:

$$\phi(x) = f(b) - f(x) - (b - x)S. \tag{5}$$

The function $\phi(x)$ is continuous in the interval (a, b) since this is true of $f(x)$. It is clear from (4) and (5) that

$$\phi(a) = 0 \quad \text{and} \quad \phi(b) = 0.$$

Further, since $\phi'(x) = -f'(x) + S$, it follows that $\phi(x)$ has a derivative at every interior point of the interval (a, b) as this is true of $f(x)$. Accordingly, the function $\phi(x)$ satisfies the conditions of Rolle's theorem. Hence there is a value x_1 of x lying between a and b such that

$$\phi'(x_1) = 0, \quad a < x_1 < b.$$

That is,

$$-f'(x_1) + S = 0$$

and

$$S = f'(x_1), \quad a < x_1 < b.$$

This proves the theorem.

168. Taylor's Theorem. *If the single-valued function $f(x)$ and its first $n - 1$ derivatives are continuous in the interval (a, b) and*

if, further, it has an nth *derivative* $f^{(n)}(x)$ *at all interior points of the interval* (a, b), *then*

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n, \quad (1)$$

where

$$R_n = \frac{(b-a)^n}{n!}f^{(n)}(x_1), \quad a < x_1 < b.$$

Define a number S by the equation:

$$f(b) - f(a) - (b - a)f'(a) - \frac{(b-a)^2}{2!}f''(a) - \cdots - \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{(b-a)^n}{n!}S = 0. \quad (2)$$

Define a function $\phi(x)$ as follows:

$$\phi(x) = f(b) - f(x) - (b - x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) - \frac{(b-x)^n}{n!}S. \quad (3)$$

From (2) and (3) it follows that $\phi(a) = 0$ and $\phi(b) = 0$. From the hypotheses made concerning the function $f(x)$ and its derivatives it is clear that $\phi(x)$ satisfies all of the remaining conditions of Rolle's theorem. Hence

$$\phi'(x_1) = 0, \quad a < x_1 < b.$$

But

$$\phi'(x) = -\frac{(b-x)^{n-1}}{(n-1)!}f^n(x) + \frac{n(b-x)^{n-1}}{n!}S.$$

Hence

$$S = f^n(x_1), \quad a < x_1 < b.$$

On entering this value of S in equation (2) it is seen that the truth of equation (1) has been established. The quantity R_n in equation (1) is called the *remainder* after n terms.

If we replace b by x in equation (1), we obtain

$$f(x) = f(a) + (x - a)f'(x) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n, \quad (4)$$

where

$$R_n = \frac{(x-a)^n}{n!}f^{(n)}(x_1), \quad a < x_1 < x.$$

This equation holds for any value x such that the conditions of Taylor's theorem are satisfied by the function $f(x)$ and its derivatives in the interval (a, x).

A form of Taylor's theorem that is of especial interest is found when $a = 0$. In this case, equation (1) becomes

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n-1)}(0)\frac{x^{n-1}}{(n-1)!} + R_n, \quad (5)$$

where

$$R_n = f^{(n)}(x_1)\frac{x^n}{n!}, \quad 0 < x_1 < x.$$

The form (5) of Taylor's theorem is known as *Maclaurin's theorem*.

169. Taylor's and Maclaurin's Series. If $f(x)$ is continuous and possesses derivatives of all orders in an interval and if $\lim_{n\to\infty} R_n = 0$, the number of terms in (4) and (5), §168, can be increased indefinitely. These equations then become respectively

$$f(x) = f(a) + f'(a)\,(x - a) + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} + \cdots. \quad (1)$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + \cdots. \quad (2)$$

In (1), $f(x)$ and its derivatives are assumed to be continuous from a to x.

In (2), $f(x)$ and its derivatives are assumed to be continuous from 0 to x.

The series (1) is called *Taylor's series* and (2) is called *Maclaurin's series.*

Illustration 1. Represent $\sin x$ by a power series in $(x - a)$. Use formula (1).

$$\begin{aligned} f(x) &= \sin x & \qquad f(a) &= \sin a \\ f'(x) &= \cos x & f'(a) &= \cos a \\ f''(x) &= -\sin x & f''(a) &= -\sin a \\ f'''(x) &= -\cos x & f'''(a) &= -\cos a \\ f^{IV}(x) &= \sin x & f^{IV}(a) &= \sin a \\ f^{V}(x) &= \cos x & f^{V}(a) &= \cos a \end{aligned}$$

Then by (1)

$$\sin x = \sin a + \cos a\,(x - a) - \sin a\,\frac{(x-a)^2}{2!} - \cos a\,\frac{(x-a)^3}{3!} + \sin a\,\frac{(x-a)^4}{4!} + \cos a\,\frac{(x-a)^5}{5!} - \cdots$$

The remainder after n terms, R_n, will be less in numerical value than $\left|\frac{(x-a)^n}{n!}\right|$ since in the expression for R_n this factor will appear multiplied by either $\cos x_1$ or $\sin x_1$ whose numerical value is less than or equal to 1. It follows readily that $\lim_{n\to\infty} R_n = 0$ for any values of x and a. Hence the power series written represents the function $\sin x$.

$\sin 33°$ will be computed by using the expansion in power series just obtained. Since the formulas for the differentiation of $\sin x$ and $\cos x$ assume that x is measured in radians, the angles must be expressed in radians. To find $\sin 33°$, take $a = 30° = \frac{\pi}{6}$, $x = 33° = \frac{11\pi}{60}$ and $x - a = 3° = \frac{\pi}{60} = 0.0524$.

$$\sin 33° = 0.5 + (0.8660)(0.0524) - \frac{0.5}{2}(0.0524)^2 - \frac{0.8660}{6}(0.0524)^3 + \frac{0.5}{24}(0.0524)^4 + \cdots$$
$$= 0.5 + 0.0454 - 0.0007 - 0.0001$$
$$= 0.5446.$$

The error made in stopping with the fourth term is given by

$$|R_4| < \frac{(0.0524)^4}{4!} < 3 \times 10^{-7}.$$

The calculations above were not carried out to a sufficient number of decimal places to secure this degree of accuracy in the result.

Illustration 2. Expand $\sin x$ in a power in x.

$$\begin{array}{rlrl} f(x) = & \sin x & f(0) = & 0 \\ f'(x) = & \cos x & f'(0) = & 1 \\ f''(x) = & -\sin x & f''(0) = & 0 \\ f'''(x) = & -\cos x & f'''(0) = & -1 \\ f^{IV}(x) = & \sin x & f^{IV}(0) = & 0 \\ f^{V}(x) = & \cos x & f^{V}(0) = & 1 \end{array}$$

Then by (2)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

The remainder after n terms clearly approaches zero as n becomes infinite. The reasoning used in *Illustration* 1 applies.

By the use of this series, the sine of a small angle can easily be found. Thus to find $\sin 6°$, substitute $x = \frac{\pi}{30} = 0.10472$.

$$\sin 6° = 0.10472 - 0.00019 = 0.10453.$$

The error made in stopping with the term in x^3 can be calculated by considering the value of R_5 since the coefficient of x^4 in the expansion is zero.

$$|R_5| < \frac{(0.10472)^5}{120} < 1.6 \times 10^{-7}.$$

The calculations above were not carried to a sufficient number of decimal places to secure this degree of accuracy in the result.

Illustration 3. Expand e^x in a power series in x and find the value of e correct to five decimal places.

$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f'(x) &= e^x & f'(0) &= 1 \\ f''(x) &= e^x & f''(0) &= 1 \end{aligned}$$

and so on. Then

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

In this case $R_n = \dfrac{e^{x_1}x^n}{n!}$ where $0 < x_1 < x$. For a definite x, no matter how large, $\lim\limits_{n\to\infty} R_n = 0$. Hence the series written represents e^x.

By letting $x = 1$ in the series for e^x we obtain the following important series:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots ,$$

which enables us to calculate the number e to any desired degree of accuracy. The computation can be conveniently arranged as follows, noting that the fourth term can be obtained from the third by dividing it by 3, the fifth from the fourth by dividing it by 4, and so on.

$$\begin{array}{r} 1.000000 \\ 1.000000 \\ 0.500000 \\ 0.166667 \\ 0.041667 \\ 0.008333 \\ 0.001389 \\ 0.000198 \\ 0.000025 \\ 0.000003 \\ \hline 2.71828 \end{array}$$

Hence

$$e = 2.71828.$$

The error committed by neglecting all terms after $\frac{1}{9!}$ is less than

$$\frac{1}{10!}\left(\frac{1}{1-\frac{1}{10}}\right) = \frac{1}{9(9!)} = 0.0000003.$$

(see *Illustration* 3, §164).

Exercises

1. Expand $\cos x$ in a power series in x.

2. Expand $\cos x$ in a power series in $(x - a)$.

3. Expand e^x in a power series in $(x - a)$.

4. Using the result of Exercise 3, find correct to four decimal places the values of $e^{1.05}$, $e^{0.97}$, $e^{2.1}$.

5. By the use of the series already found, compute:

(*a*) $\sqrt[3]{e}$ to five decimal places.

(*b*) $\sqrt[10]{e}$ to six decimal places.

(*c*) $\sin 3°$ to six decimal places.

(*d*) cosine of one radian to four decimal places.

6. By the use of the result of Exercise 2, find $\cos 33°$ correct to four decimal places.

7. Expand $\log (1 + x)$ in a power series in x.

8. Expand $\log (1 - x)$ in a power series in x.

9. Find $\sin 32°$ correct to four decimal places.

10. For what values of x are the series of Exercises 1, 7, and 8 and of *Illustrations* 2 and 3 convergent.

11. Expand $\tan^{-1} x$ in a power series in x.

HINT. $f'(x) = \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots$

12. Expand $\sin^{-1} x$ in a power series in x.

13. Expand $(1 + x)^n$ in a power series in x.

14. Expand $e^{\sin x}$ in a power series in x as far as the term containing x^4.

15. Expand $e^{\cos x}$ in a power series in x.

16. Expand $e^x \sin x$ in a power series in x.

17. Expand $e^x \cos x$ in a power series in x.

18. Expand $\sin mx$ in a power series in x.
19. Expand $\tan x$ in a power series in x.
20. Expand $\sinh x$ in a power series in x. Also obtain result from the relation $\sinh x = \frac{1}{2}(e^x - e^{-x})$, using the series for e^x and e^{-x}.
21. Expand $\cosh x$ in a power series in x.

170. Taylor's and Maclaurin's Series. Second Form of Proof. The representation of a function by a power series in $x - a$ (see §165) will be obtained first. The representation by a power series in x will then follow as a special case.

Certain assumptions are made in the derivation of the general formula and no attempt is made to justify them. A more rigorous derivation of the same results has been given in §§166–169.

Assume that $f(x)$ can be represented by a power series in $(x - a)$:

$$f(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots, \tag{1}$$

where $a_0, a_1, a_2, \ldots, a_n, \ldots$ are coefficients which are to be determined. Assume further that the result of differentiating the second member term by term any given number of times is equal to the corresponding derivative of the first member. Then,

$$\left.\begin{aligned} f'(x) &= a_1 + 2a_2(x - a) + 3a_3(x - a)^2 \\ &\qquad + \cdots + na_n(x - a)^{n-1} + \cdots \\ f''(x) &= 2a_2 + 6a_3(x - a) \\ &\qquad + \cdots + n(n-1)a_n(x - a)^{n-2} + \cdots \\ f'''(x) &= 6a_3 + \cdots + n(n-1)(n-2)a_n(x - a)^{n-3} + \cdots \\ &\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f^{(n)}(x) &= (n!)a_n + \cdots \end{aligned}\right] . \tag{2}$$

Put $x = a$ in (1) and (2).

$$\begin{aligned} f(a) &= a_0, \\ f'(a) &= a_1, \\ f''(a) &= 2a_2, \\ f'''(a) &= (3!)a_3 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f^{(n)}(a) &= (n!)a_n. \end{aligned}$$

Whence

$$a_0 = f(a),$$
$$a_1 = f'(a),$$
$$a_2 = \frac{f''(a)}{2!},$$
$$a_3 = \frac{f'''(a)}{3!},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_n = \frac{f^{(n)}(a)}{n!},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Substituting in (1) we obtain

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots. \tag{3}$$

This series is known as *Taylor's series* representing the function $f(x)$.

A power series in x representing $f(x)$ is obtained by letting $a = 0$ in (3). It follows that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots. \tag{4}$$

This series is called *Maclaurin's series* representing the function $f(x)$.

171. Computation of Logarithms. The series of Exercise 7, **§169,** for $\log(1 + x)$ is convergent only when $-1 < x \leqq +1$, and that for $\log(1 - x)$, Exercise 8, **§169,** only when $-1 \leqq x < +1$. It would appear then impossible to find the logarithm of a number greater than 2 by these formulas. By a very simple device it is, however, possible to obtain formulas for finding the logarithm of any number.

From the series

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots$$
$$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \cdots$$

it follows that

$$\log \frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right], \tag{1}$$

where $|x| < 1$. Let $x = \dfrac{1}{2z+1}$. Then

$$\frac{1+x}{1-x} = \frac{z+1}{z}$$

and

$$\log \frac{z+1}{z} = 2\left[\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right], \tag{2}$$

where $z > 0$, or

$$\log (z+1) = \log z + 2\left[\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right]. \tag{3}$$

By letting $z = 1$, log 2 can be computed by this formula. The series is much more rapidly convergent than that for $\log (1 + x)$ with $x = 1$. In fact, 200 terms of the latter series must be taken to obtain log 2 correct to two decimal places, while four terms of the new series (3) will give log 2 correct to four decimal places. After log 2 has been found, log 3 can be found by setting $z = 2$. The logarithm of 4 is found by taking twice log 2; log 5 by setting $z = 4$; log 6 by adding log 3 and log 2, and so on.

Exercise

Compute log 5 correct to four decimal places, given that log 4 $= 1.38629$. Here, as always in the calculus, the base is understood to be e.

172. Computation of π. By letting $x = 1$ in the series for $\tan^{-1} x$, Exercise 11, §169, the following equation is obtained from which π can be computed:

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots.$$

This series converges very slowly. To obtain a more rapidly con-

verging series make use of the relation

$$\tan^{-1} 1 = \tan^{-1} \tfrac{1}{2} + \tan^{-1} \tfrac{1}{3}.$$

Then

$$\frac{\pi}{4} = \tfrac{1}{2} - \frac{1}{(3)(2^3)} + \frac{1}{(5)(2^5)} - \frac{1}{(7)(2^7)} + \cdots$$
$$+ \tfrac{1}{3} - \frac{1}{(3)(3^3)} + \frac{2}{(5)(3^5)} - \frac{1}{(7)(3^7)} + \cdots.$$

173. Relation between the Exponential and Circular Functions. If it be admitted that the expansion in Maclaurin's series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \tag{1}$$

which was proved for real values of z, is also true when z is imaginary, we obtain, on setting $z = ix$,

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots$$
$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots. \tag{2}$$

On separating real and imaginary parts this becomes

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right). \tag{3}$$

Since (Exercise 1 and *Illustration* 2, §169),

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

it follows that

$$\mathbf{e^{ix} = \cos x + i \sin x.} \tag{4}$$

On changing the sign of x it results that

$$e^{-ix} = \cos x - i \sin x. \tag{5}$$

Solving equations (4) and (5) for $\cos x$ and $\sin x$,

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{6}$$

and

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{7}$$

These interesting relations between the circular and exponential functions are of very great importance.

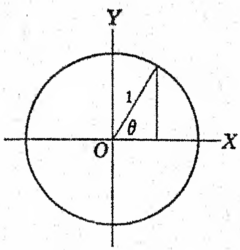

FIG. 129.

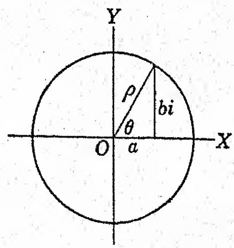

FIG. 130.

If θ represent the vectorial angle in the complex number plane, then it is clear from Fig. 129 that $e^{i\theta}$ represents a point on the unit circle (circle of radius 1 about the origin as center) in this plane. Further, any complex number $a + bi$ can be put in the form $\rho e^{i\theta}$, for (Fig. 130)

$$a + bi = \rho(\cos \theta + i \sin \theta) = \rho e^{i\theta},$$

where $\rho = \sqrt{a^2 + b^2}$.

Exercises

Represent by a point in the complex plane:

1. $3e^{\frac{i\pi}{3}}$.
2. $2e^{-\frac{i\pi}{3}}$.
3. $e^{\frac{i\pi}{4}}$.
4. $e^{\frac{i\pi}{2}}$.
5. $e^{i\pi}$.
6. $e^{-i\pi}$.
7. $e^{2i\pi}$.
8. $5e^{\frac{3i\pi}{4}}$.
9. Express the numbers of Exercises 1–8 in the form $a + bi$.

174. DeMoivre's Theorem. The interesting and important theorem, known as DeMoivre's theorem,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \tag{1}$$

can be easily established by the use of the relation (4) of §173. For

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

Exercises

Find, by the use of (1):

1. The cube of $1 + i$.
2. The square of $\dfrac{1 + i\sqrt{3}}{2}$.
3. The cube of $\dfrac{1 + i\sqrt{3}}{2}$.
4. The cube of $\dfrac{-1 + i\sqrt{3}}{2}$.
5. The cube of $\dfrac{-1 - i\sqrt{3}}{2}$.
6. The cube of $\dfrac{5(1 + i\sqrt{3})}{2}$.

In (1), n may be a fraction as well as an integer. It will then indicate a root instead of a power. In this case we do not have simply one root,

$$(\cos\theta + i\sin\theta)^{\frac{1}{m}} = \cos\frac{\theta}{m} + i\sin\frac{\theta}{m},$$

(n having been placed equal to $\frac{1}{m}$, where m is an integer), but $m - 1$ additional roots. This follows from the fact that

$$e^{i\theta} = e^{i(\theta + 2p\pi)}, \tag{2}$$

where $p = 0, 1, 2, 3, 4, \cdots, m, m + 1, \cdots$. Hence we can write

$$(\cos\theta + i\sin\theta)^{\frac{1}{m}} = [e^{i\theta}]^{\frac{1}{m}} = [e^{i(\theta+2p\pi)}]^{\frac{1}{m}},$$

or

$$(\cos\theta + i\sin\theta)^{\frac{1}{m}} = e^{\frac{i(\theta+2p\pi)}{m}}, \quad (p = 0, 1, 2, \cdots). \tag{3}$$

It would appear at first sight as if there were infinitely many roots corresponding to the infinitely many values of p. But a little consideration shows that when $p \geqq m$, the roots already found by letting p take the values $0, 1, 2, \cdots, m-1$, repeat themselves, since $e^{2i\pi} = 1$. There are then exactly m m^{th} roots of $e^{i\theta} = \cos\theta + i\sin\theta$,

$$e^{\frac{i(\theta+2p\pi)}{m}} = \cos\frac{\theta+2p\pi}{m} + i\sin\frac{\theta+2p\pi}{m}, \tag{4}$$

where $p = 0, 1, 2, \cdots, m-1$.

Illustration. Find the three cube roots of -1.

$$\begin{aligned}(-1)^{\frac{1}{3}} &= (e^{i\pi})^{\frac{1}{3}} \\ &= [e^{i(\pi+2p\pi)}]^{\frac{1}{3}} && (p = 0, 1, 2) \\ &= e^{\frac{i(\pi+2p\pi)}{3}} && (p = 0, 1, 2) \\ &= e^{\frac{i\pi}{3}},\ e^{i\pi},\ \text{and } e^{\frac{5i\pi}{3}}.\end{aligned}$$

Exercises

1. Show that the three cube roots of $a + bi = \rho e^{i\theta}$ are: $\sqrt[3]{\rho}e^{\frac{i\theta}{3}}$, $\sqrt[3]{\rho}e^{\frac{i(\theta+2\pi)}{3}}$, and $p\sqrt[3]{\rho}e^{\frac{i(\theta+4\pi)}{3}}$. How would these roots be determined graphically?

2. Find the two square roots of $1 + i$.

3. Find graphically the two square roots of i.

4. Find graphically the three cube roots of 1.

175. Indeterminate Forms. The Form $\frac{0}{0}$. If $f(x)$ and $\phi(x)$ are two functions of x and if $f(a) = 0$ and $\phi(a) = 0$ the value of the quotient $\dfrac{f(x)}{\phi(x)}$ at $x = a$ is defined as $\lim\limits_{x\to a}\dfrac{f(x)}{\phi(x)}$ if this limit exists. This limit can be calculated in many instances by special devices. See, for example, §§**26**, **55**, **56**, and **57**. It will be observed that finding the derivative of a function involves the calculation of such a limit since by definition $\dfrac{dy}{dx} = \lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$. In many instances the

limit in question can best be determined with the aid of the following theorem:

Cauchy's Formula. *If $f(x)$ and $\phi(x)$ are continuous in the interval (a, b), if each function has a derivative at all interior points of the interval, and if $\phi'(x)$ does not vanish at any interior point of the interval, then*

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(x_1)}{\phi'(x_1)}$$

where $a < x_1 < b$.

Form the function

$$\psi(x) = \frac{f(b) - f(a)}{\phi(b) - \phi(a)}[\phi(x) - \phi(a)] - [f(x) - f(a)].$$

From the fact that $\psi(b) = 0$ and $\psi(a) = 0$ and from the assumptions concerning $f'(x)$ and $\phi'(x)$ it follows that the conditions of Rolle's theorem are satisfied by $\psi(x)$. Hence

$$\psi'(x_1) = \frac{f(b) - f(a)}{\phi(b) - \phi(a)}\phi'(x_1) - f'(x_1) = 0,$$

$a < x_1 < b$. Consequently

$$\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(x_1)}{\phi'(x_1)}, \quad a < x_1 < b.$$

We can now calculate the limit referred to above, *viz.*,

$$\lim_{x\to a} \frac{f(x)}{\phi(x)}$$

when $f(a) = 0$, $\phi(a) = 0$.

To do so we have only to replace b by x in the theorem just proved. We obtain

$$\frac{f(x)}{\phi(x)} = \frac{f'(x_1)}{\phi'(x_1)}, \quad a < x_1 < x.$$

Then

$$\lim_{\mathbf{x}\to \mathbf{a}} \frac{\mathbf{f}(\mathbf{x})}{\phi(\mathbf{x})} = \lim_{\mathbf{x}\to \mathbf{a}} \frac{\mathbf{f}'(\mathbf{x})}{\phi'(\mathbf{x})}$$

if this limit exists.

If $f'(a)$ and $\phi'(a)$ are also both zero

$$\lim_{x\to a}\frac{f(x)}{\phi(x)} = \lim_{x\to a}\frac{f'(x)}{\phi'(x)} = \lim_{x\to a}\frac{f''(x)}{\phi''(x)},$$

and so on.

Illustration 1.

$$\lim_{x\to 0}\frac{e^x - 1}{x} = \lim_{x\to 0}\frac{e^x}{1} = 1.$$

Illustration 2.

$$\begin{aligned}\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x - \sin x} &= \lim_{x\to 0}\frac{e^x + e^{-x} - 2}{1 - \cos x}\\ &= \lim_{x\to 0}\frac{e^x - e^{-x}}{\sin x}\\ &= \lim_{x\to 0}\frac{e^x + e^{-x}}{\cos x}\\ &= 2.\end{aligned}$$

The Form $\frac{\infty}{\infty}$. If $f(a)$ and $\phi(a)$ are both infinite, it will be shown that

$$\lim_{x\to a}\frac{f(x)}{\phi(x)} = \lim_{x\to a}\frac{f'(x)}{\phi'(x)} = L$$

if this latter limit exists. Consider two values of x, x_0 and x in the vicinity of $x = a$ such that $a < x < x_0$. Now

$$\frac{f(x) - f(x_0)}{\phi(x) - \phi(x_0)} = \frac{f(x)}{\phi(x)}\,\frac{1 - \dfrac{f(x_0)}{f(x)}}{1 - \dfrac{\phi(x_0)}{\phi(x)}}. \tag{1}$$

But by Cauchy's formula

$$\frac{f(x) - f(x_0)}{\phi(x) - \phi(x_0)} = \frac{f'(x_1)}{\phi'(x_1)}, \quad x < x_1 < x_0.$$

Then

$$\frac{f'(x_1)}{\phi'(x_1)} = \frac{f(x)}{\phi(x)}\,\frac{1 - \dfrac{f(x_0)}{f(x)}}{1 - \dfrac{\phi(x_0)}{\phi(x)}}. \tag{2}$$

By choosing x_0 sufficiently near to a we can make the left-hand member differ from $L = \lim\limits_{x\to a} \dfrac{f'(x)}{\phi'(x)}$ by as little as we please. Then x_0 having been chosen x can be taken so much nearer a that the second factor in the right-hand member of (2) differs from 1 by as little as we please. Hence

$$\lim_{x\to a} \frac{f(x)}{\phi(x)} = L = \lim_{x\to a} \frac{f'(x)}{\phi'(x)}.$$

If a is infinite the preceding argument is to be modified by choosing x_0 very large and making suitable minor changes in the subsequent steps. The conclusion is that

$$\lim_{x\to\infty} \frac{f(x)}{\phi(x)} = \lim_{x\to\infty} \frac{f'(x)}{\phi'(x)}$$

provided that this latter limit exists.

Illustration 3.

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

The Form $0 \cdot \infty$. The indeterminate form $0 \cdot \infty$ can be thrown into either of the forms $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$. Thus

$$\lim_{x\to 0} x \cot x = \lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{1}{\sec^2 x} = 1.$$

Other indeterminate forms are $\infty - \infty$, 1^∞, 0^0, ∞^0. Their evaluation can be made to depend upon that of one of the preceding forms. Thus

$$\lim_{x\to\frac{\pi}{2}} (\tan x - \sec x) = \lim_{x\to\frac{\pi}{2}} \frac{\sin x - 1}{\cos x}.$$

Again, to find

$$\lim_{x\to 0} x^{\sin x},$$

let $y = x^{\sin x}$.

$$\log y = \sin x \log x$$

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log x}{\csc x} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\csc x \cot x}$$

$$= -\lim_{x\to 0} \frac{\sin^2 x}{x \cos x} = 0.$$

Hence

$$\lim_{x\to 0} x^{\sin x} = \lim_{x\to 0} y = e^0 = 1.$$

Exercises

Evaluate the following:

1. $\lim\limits_{x\to 1} \dfrac{\log x}{x-1}$.

2. $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\cos\theta \sin^2\theta}$.

3. $\lim\limits_{x\to 0} \dfrac{x\cos x - \sin x}{x}$.

4. $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x - \sin x}$.

5. $\lim\limits_{x\to 1} \dfrac{x^n - 1}{x-1}$.

6. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\sin 2x}$.

7. $\lim\limits_{x\to 0} \dfrac{\tan 3x}{x}$.

8. $\lim\limits_{x\to 0} \dfrac{x^3}{\sin x - x}$.

9. $\lim\limits_{x\to 3} \dfrac{x^2 - x - 6}{x^2 - 9}$.

10. $\lim\limits_{x\to \infty} \dfrac{3x^2+5}{4x^2+1}$.

11. $\lim\limits_{x\to 0} \dfrac{\log x}{\cot x}$.

12. $\lim\limits_{\theta\to \frac{\pi}{2}} \dfrac{\tan 2\phi}{\tan 5\phi}$.

13. $\lim\limits_{x\to \infty} \dfrac{x^n}{\log x}$.

14. $\lim\limits_{x\to 0} \dfrac{x^n}{e^x}$.

15. $\lim\limits_{x\to \infty} \dfrac{e^x}{x^3}$.

16. $\lim\limits_{x\to \infty} e^x \tan \dfrac{1}{x}$.

17. $\lim\limits_{x\to 1} \left[\dfrac{2}{x^2-1} - \dfrac{1}{x-1}\right]$.

18. $\lim\limits_{\theta\to \frac{\pi}{2}} \left[\left(\dfrac{\pi}{2} - \theta\right) \tan\theta\right]$.

19. $\lim\limits_{x\to 1} \left[\dfrac{1}{\log x} - \dfrac{1}{x-1}\right]$.

20. $\lim\limits_{x\to 0} (\cos x)^{\cot x}$.

21. $\lim\limits_{x\to 0} (1-x)^{\frac{1}{x}}$.

22. $\lim\limits_{x\to 0} (\csc x)^{\tan x}$.

23. $\lim\limits_{x\to 0} (\sin x)^{\tan x}$.

24. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$.

25. $\lim\limits_{\theta\to 0} \dfrac{\tan\theta}{\sin\theta}$.

CHAPTER XVII

TOTAL DERIVATIVE. EXACT DIFFERENTIAL

176. The Total Derivative. Let $z = f(x, y)$ and let x and y be functions of a third variable t, the time, for example. We seek an expression for $\frac{dz}{dt}$, the derivative of z with respect to t, in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

As an illustration of what is meant, let z denote the area of a rectangle whose sides x and y are functions of t, and at a given instant let each side be changing at a certain rate. The rate at which the area is changing is sought.

Returning to the general problem, let t take on an increment Δt. Then x takes on the increment Δx and y the increment Δy, and consequently z the increment Δz. We then have

$$z = f(x, y) \tag{1}$$

$$z + \Delta z = f(x + \Delta x, y + \Delta y)$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \tag{2}$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y) \tag{3}$$

$$\frac{\Delta z}{\Delta t} = \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x}\frac{\Delta x}{\Delta t} + \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}\frac{\Delta y}{\Delta t}. \tag{4}$$

Taking the limits of both sides of (4) as Δt approaches zero, we have

$$\frac{dz}{dt} = \frac{\partial f(x, y)}{\partial x}\frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dt}, \tag{5}$$

since Δx and Δy approach zero as Δt approaches zero. Equation

(5) can be written in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{6}$$

This states that the rate of change of z with respect to t is equal to the rate of change of z with respect to x, times the rate of change of x with respect to t, plus the rate of change of z with respect to y, times the rate of change of y with respect to t.

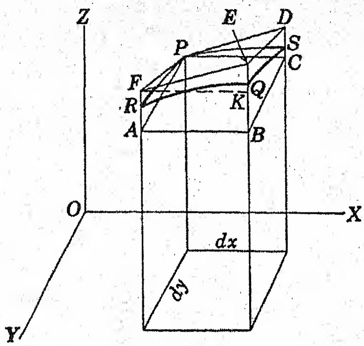

FIG. 131.

If $t = x$, (6) becomes

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}.$$

This formula applies when $z = f(x, y)$ and y is a function of x, *e.g.*, $y = \phi(x)$.

Multiplying (6) by dt we obtain

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy. \tag{7}$$

This defines dz, which is called the *total differential of* z.

We shall now give a geometrical interpretation of dz. Let P, Fig. 131, be the point (x, y, z) on the surface $z = f(x, y)$. Let

$$PC = dx$$

and

$$PA = dy.$$

Then Q is the point $(x + dx,\ y + dy,\ z + \Delta z)$. Let $PDEF$ be the plane tangent to the surface at the point P. Then PF is tangent to the arc PR, and PD is tangent to the arc PS.

From F draw FK parallel to AB meeting BE in K.

$$BE = BK + KE$$
$$BK = AF = \frac{\partial z}{\partial y}\,dy.$$

Since $FK = PC$ and $PD = FE$, triangle KFE is equal to the triangle CPD, and

$$KE = CD = \frac{\partial z}{\partial x}\,dx.$$

Therefore

$$BE = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

Hence $BE = dz$. Consequently, dz may be interpreted as the increment measured to the tangent plane to $z = f(x, y)$ at the point $P\ (x, y, z)$ when x and y are given the increments dx and dy respectively.

If dx and dy are small, dz is approximately equal to Δz, just as in the case of a function of a single independent variable, $y = f(x)$, dy is approximately equal to Δy if dx is small. Accordingly, the relation (7) can be used to calculate the approximate value of Δz if dx and dy are small.

If z is a function of three or more independent variables, relations corresponding to (6) and (7) hold. Thus, if $z = f(x, y, u)$,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \frac{\partial z}{\partial u}\frac{du}{dt}.$$

Illustration 1. If $z = xy$, the area of a rectangle of sides x and y, we obtain, by using (7),

$$dz = y\,dx + x\,dy.$$

The first term on the right-hand side represents the area of the strip *BEFC*, Fig. 132. The second term represents the area of *DCGH*. The difference between Δz and dz is the area of the rectangle *CFLG*, which becomes relatively smaller, the smaller dx and dy become.

If the sides of a rectangle are measured and found to be 10 and 6 feet with a possible error of 0.2 and 0.1 foot, respectively, the approximate possible error in the area as computed from these measurements is given by

$$dz = (6)(0.2) + (10)(0.1) = 2.2.$$

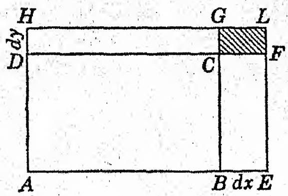

FIG. 132.

The approximate possible error is 2.2 square feet. The area lies between the approximate limits of 62.2 and 57.8 square feet. The corresponding accurate limits are 62.22 and 57.82 square feet.

Illustration 2. The base of a rectangular piece of brass is 15 feet and its altitude is 10 feet. If the base is increasing in length at the rate of 0.03 foot per hour and the altitude at the rate of 0.02 foot per hour, at what rate is the area changing?

Let x denote the base, y the altitude, and z the area. Then

$$z = xy$$

and
$$\frac{dz}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$$
$$= (10)(0.03) + (15)(0.02).$$

Illustration 3. $z = \dfrac{x}{y}$.

$$\frac{\partial z}{\partial x} = \frac{1}{y}, \qquad \frac{\partial z}{\partial y} = -\frac{x}{y^2}.$$

and, by (7)

$$dz = \frac{1}{y}\,dx - \frac{x}{y^2}\,dy,$$

or

$$dz = \frac{y\,dx - x\,dy}{y^2}.$$

Exercises

Find the total differential of each of the following functions:

1. $z = x^2y^3$.
2. $z = e^x \sin y + y^2$.
3. $z = \frac{x^2}{y}$.
4. $z = \sqrt{x^2 + y^2}$.
5. $u = \sqrt{x^2 + y^2 + z^2}$.
6. $z = x \sin y$.
7. $z = x \cos y$.
8. $z = \tan y$.
9. $z = e^x \cos y + x^2 + y^2$.
10. $z = e^y \sin x + y^2$.
11. $z = e^{-ax} \cos ny$.
12. $z = \frac{x}{y} + x$.
13. $z = x^2 \log y$.
14. $v = \pi r^2 h$.

Find $\frac{dz}{dt}$ if

15. $z = x^2y$.
16. $z = \pi x^2 y$.
17. $z = \sqrt{x^2 + y^2}$.
18. $z = xyu$.
19. $z = c\frac{x}{y}$.

20. The radius of the base of a right circular cylinder is 5 inches and its altitude is 15 inches. If the radius of the base is increasing at the rate of 0.2 inch per minute and the altitude at the rate of 0.4 inch per minute, at what rate is the volume increasing?

21. The radius of a right circular cylinder is 10 inches and its altitude is 25 inches. If the radius of the base and the altitude are each increased by 0.2 inch, by how much, approximately, is the volume increased?

22. A closed cylindrical tank is 2 feet in diameter and 6 feet high, inside dimensions. Approximately how much metal is in the walls and ends of the tank if they are 0.25 inch thick?

23. The hypotenuse and an acute angle of a right triangle are measured and found to be 14.2 feet and 35°, respectively. Find the approximate possible error in the computed length of the side adjacent to the measured angle if the length of the hypotenuse may be in error by 0.1 foot and the angle by 0.5°.

24. The angle of elevation of the top of a vertical cliff is found to be 28° with a possible error of 0.5°. The distance to the base of the cliff measured along the level ground is found to be 800 feet with a possible error of 0.2 foot. Find the possible error in the height of the cliff as computed from these measurements.

25. The legs of a right triangle are found to be 10.4 and 8.3 feet with a possible error of 0.1 foot in each measurement. Find, approximately, the possible error in the hypotenuse as computed from these lengths.

26. Find the possible error in the length of the hypotenuse of the triangle of Exercise 25 if, in addition to the possible errors in the measurement of the legs, the angle included by them may differ from a right angle by 0.5°.

27. The formula connecting the pressure, volume, and absolute temperature of a perfect gas is $pv = BT$, B being a constant. If $T = 450°$, $p = 5000$ pounds per square foot, and $v = 18.4$ cubic feet, find the approximate change in p when T changes to 452° and v to 18.6 cubic feet.

28. If, with the data of Exercise 27, the temperature is changing at the rate of 1° per minute and the volume at the rate of 0.4 cubic foot per minute, at what rate is the pressure changing?

29. The constant C in Boyle's law, $pv = C$, is to be found by measuring p and v. If p is measured and found to be 10,000 pounds per square foot with a possible error of 100 pounds per square foot, and v is found to be 20.4 cubic feet with a possible error of 0.4 cubic foot, find the approximate possible error in C as computed from these measurements.

30. The period of a simple pendulum is given by the formula

$$T = \pi\sqrt{\frac{l}{g}}.$$

g at the place of observation is known to be 980.6 centimeters per second per second with a possible error of 0.1 centimeter per second per second, and l is measured and found to be 102 centimeters with a possible error of 0.1 centimeter. Find the approximate possible error in T.

177. Exact Differential. An expression of the form

$$M\,dx + N\,dy,$$

where M and N are functions of x and y, may or may not be the differential of some function of x and y. If it is, it is called an *exact differential.* Thus

$$\sin y\,dx + x\cos y\,dy \tag{1}$$

is an exact differential, for it is the differential of $z = x \sin y$. The coefficient of dx is $\frac{\partial z}{\partial x} = \sin y$, and that of dy is $\frac{\partial z}{\partial y} = x \cos y$.

$$x^2 \sin y\,dx + x\cos y\,dy \tag{2}$$

is not an exact differential. It is fairly evident from (1) that we cannot find a function $z = f(x, y)$ such that $\frac{\partial z}{\partial x} = x^2 \sin y$ and $\frac{\partial z}{\partial y} = x \cos y$.

We seek a test for determining whether or not an expression of the form

$$M\,dx + N\,dy \tag{3}$$

is an exact differential. If (3) is the exact differential of a function z, we must have

$$\frac{\partial z}{\partial x} = M \tag{4}$$

and

$$\frac{\partial z}{\partial y} = N, \tag{5}$$

since

$$dz = \frac{\partial z}{dx}dx + \frac{\partial z}{\partial y}dy. \tag{6}$$

Differentiate (4) with respect to y and (5) with respect to x and obtain

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial M}{\partial y} \tag{7}$$

and

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial N}{\partial x}. \tag{8}$$

Since, in general,

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y},$$

it follows that, if (3) is an exact differential, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{9}$$

The condition (9) must be satisfied if (3) is an exact differential. It does not follow, however, without further proof, that (3) is an exact differential if (9) is satisfied. It can, however, be shown that this is the case. The proof will be omitted. The expression (3) cannot be an exact differential unless (9) is satisfied and is an exact differential if (9) is satisfied.

When an expression of the form (3) is given, the first step is to determine whether or not it is an exact differential by applying the test (9). If it is an exact differential, the next step is to find the function z of which it is the differential. This step will be illustrated by integrating several differentials for which the functions from which they were obtained by differentiation are known.

Illustration 1. If $z = x^3 + 2x^2y + y^2 + C$,

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= (3x^2 + 4xy)dx + (2x^2 + 2y)dy. \end{aligned}$$

If, then, we are given the exact differential

$$dz = (3x^2 + 4xy)dx + (2x^2 + 2y)dy$$

and are required to find the function of which it is the differential, we note first that

$$\frac{\partial z}{\partial x} = 3x^2 + 4xy.$$

Then

$$z = x^3 + 2x^2y + \textit{a function of } y \textit{ alone}.$$

And this function of y is to be so determined that

$$\frac{\partial z}{\partial y} = 2x^2 + 2y.$$

Clearly, the term $2x^2$ is obtained by taking the derivative with respect to y of $2x^2y$, a term already found. $2y$ is the derivative of y^2. y^2 is then the function of y which is to be added to the terms already found. Further an arbitrary constant is to be added since its differential will be zero. Then

$$z = x^3 + 2x^2y + y^2 + C$$

is the function whose differential was given. If, as is usually the case, it had been given that

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0, \tag{10}$$

it would have been required to find a function of x and y such that its differential would be zero. Now the first member is, as we have seen, the differential of

$$z = x^3 + 2x^2y + y^2.$$

But, if $dz = 0$, $z = C$. Then

$$x^3 + 2x^2y + y^2 = C$$

is the relation between x and y which satisfies the given equation.

Illustration 2. If

$$z = e^x \cos y + x^2 + \sin y + y^3,$$
$$dz = (e^x \cos y + 2x)\, dx + (-e^x \sin y + \cos y + 3y^2)\, dy.$$

Now let it be given that

$$(e^x \cos y + 2x)\, dx + (-e^x \sin y + \cos y + 3y^2)\, dy = 0. \tag{11}$$

From its derivation we know that the left-hand member is an exact differential, dz. Let us proceed to find z as if it were unknown.

$$\frac{\partial z}{\partial x} = e^x \cos y + 2x.$$

Then

$$z = e^x \cos y + x^2 + \textit{a function of } y \textit{ alone.} \tag{12}$$

The function of y is to be so determined that

$$\frac{\partial z}{\partial y} = -e^x \sin y + \cos y + 3y^2. \tag{13}$$

The first term is evidently obtained by differentiating $e^x \cos y$, a term already found in (12). The remaining two terms in (13) are obtained by differentiating $\sin y + y^3$. These are to be added to the terms already found in (12).
Then

$$z = e^x \cos y + x^2 + \sin y + y^3.$$

But, since $dz = 0$, $z = C$. Hence

$$e^x \cos y + x^2 + \sin y + y^3 = C$$

is a solution of (2).

Illustration 3. Integrate if possible the equation

$$(e^x y + \sin y + 2x)dx + (e^x + x \cos y + e^y + 2y - \sin y)dy = 0. \tag{14}$$

We have first to determine whether or not the first member is an exact differential. Apply the test (9).

$$\frac{\partial M}{\partial y} = e^x + \cos y.$$
$$\frac{\partial N}{\partial x} = e^x + \cos y.$$

Hence (9) is satisfied and the first member of (14) is an exact differential. On integrating the coefficient of dx with respect to x we obtain

$$e^x y + x \sin y + x^2.$$

To this we have to add

$$e^y + y^2 + \cos y,$$

the terms which arise from the integration of the coefficient of dy and which contain y alone. (The other terms in the coefficient of dy arise from the differentiation of terms already found by integrating the coefficient of dx.) Then the solution of (14) is

$$e^x y + x \sin y + x^2 + e^y + y^2 + \cos y = C.$$

178. Exact Differential Equations. *Equations involving differentials or derivatives are called differential equations.* Those of the type

$$M\,dx + N\,dy = 0, \tag{1}$$

where the first member is an exact differential, are called *exact differential equations.*

The equations (10), (11), and (14) of *Illustrations* 1, 2, and 3, **§177,** are exact differential equations. The process of finding the relation between y and x, which when differentiated gives a certain differential equation, is called the integration of the equation.

The procedure in dealing with an equation of type (1) is to determine first whether or not it is exact by applying the test (9), **§177.** If it is, integrate the coefficient of dx with respect to x and to this result add those terms which contain y only, which are obtained by integrating the coefficient of dy with respect to y.

Exercises

Are the following differential equations exact? Integrate those which are exact.

1. $3x^2y^2\,dx + 2x^3y\,dy = 0.$

2. $\dfrac{1}{y}\cos\left(\dfrac{x}{y}\right)dx - \dfrac{x}{y^2}\cos\left(\dfrac{x}{y}\right)dy = 0.$

3. $y\,e^{xy}(1 + x + y)\,dx + x\,e^{xy}(1 + x + y)\,dy = 0.$

4. $y\,e^{xy}\,dx + x\,e^{xy}\,dy = 0.$

5. $(x^3y + 2x)\,dx - (3x^2y - 5x)\,dy = 0.$

6. $\left(\dfrac{2x}{y^3} + 1\right)dx - \left(\dfrac{3x^2}{y^4} + 2y\right)dy = 0.$

7. $e^{\frac{x}{y}}\left(2 + \dfrac{x}{y}\right)dx - \dfrac{x}{y^2}\,e^{\frac{x}{y}}\left(2 + \dfrac{x}{y}\right)dy = 0.$

179. In **§154** the envelope of a family of curves was defined, and its parametric equations were found to be

$$f(x, y, c) = 0. \tag{1}$$

$$\frac{\partial f}{\partial c} = 0. \tag{2}$$

We shall now show that the envelope is tangent to each curve of the family of curves (1).

At a given point (x, y) of the curve determined by giving c a particular value in (1), the slope of the tangent is found from the equation

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

If the point also lies upon the envelope, its coordinates satisfy (1) and (2). The equation of the envelope may be regarded as given by (1) where c is the function of x and y found by solving (2) for c. On differentiating (1) with respect to x, regarding c as a function of x and y, the slope, $\frac{dy}{dx}$, of the tangent to the envelope is given by

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial c}\frac{dc}{dx} = 0, \tag{4}$$

where

$$\frac{dc}{dx} = \frac{\partial c}{\partial x} + \frac{\partial c}{\partial y}\frac{dy}{dx}.$$

But on the envelope $\frac{\partial f}{\partial c} = 0$. Hence (4) becomes

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Equations (3) and (5) show that the slope of the tangent line to the envelope at the point (x, y) is the same as the slope of the tangent line at the same point to a curve of the family (1). Hence the envelope is tangent to each curve of the family of curves (1).

CHAPTER XVIII

DIFFERENTIAL EQUATIONS

180. Differential Equations. An equation containing derivatives or differentials is called a *differential equation*. If no derivative higher than the first appears, the differential equation is said to be of the *first order*. If the equation contains the second, but no higher derivative, the equation is said to be of the *second order*. And so on. Numerous differential equations have already occurred in this course. We shall now consider the solution of differential equations somewhat systematically.

181. General Solution. Particular Integral. Let

$$f(x, y, c) = 0 \tag{1}$$

be any equation connecting x, y, and the constant c. If (1) is differentiated with respect to x, there results the equation

$$F(x, y, y', c) = 0. \tag{2}$$

Between (1) and (2) the constant c can be eliminated, giving the differential equation of a first order

$$\phi(x, y, y') = 0. \tag{3}$$

Let

$$f(x, y, c_1, c_2) = 0 \tag{4}$$

be an equation involving two constants, c_1 and c_2. By differentiating (4) twice we obtain

$$F(x, y, y', c_1, c_2) = 0 \tag{5}$$

and

$$\phi(x, y, y', y'', c_1, c_2) = 0. \tag{6}$$

Between equations (4), (5), and (6), c_1 and c_2 may be eliminated, giving a differential equation of the second order

$$\psi(x, y, y', y'') = 0. \tag{7}$$

From the equation (1) containing one arbitrary constant the differential equation of the first order (3) is obtained. From the equation (4) containing two arbitrary constants the differential equation of the second order (7) is obtained. In like manner from a relation between x and y containing n arbitrary constants a differential equation of the nth order is obtained by differentiating, and eliminating the constants.

Equation (1) is a solution of equation (3). It is called the *general solution* and involves one arbitrary constant of integration, c. Equation (4) is called the *general solution* of (7). It involves two arbitrary constants, or constants of integration. It can be shown that the general solution, or general integral, of a differential equation contains a number of arbitrary constants, or constants of integration, equal to the order of the differential equation.

A *particular integral* is obtained from the general integral by giving particular values to the constants of integration.

182. Exact Differential Equations. This type of differential equation was discussed in §178.

183. Differential Equations; Variables Separable. The variables x and y are said to be separable in a differential equation which can be put in the form $f(x)\,dx + \phi(y)\,dy = 0$. The first member is equal to a function of x alone multiplied by dx plus a function of y alone multiplied by dy.

Illustration 1.

$$(1 + y^2)x\,dx + (1 + x^2)y\,dy = 0.$$

On dividing by $(1 + y^2)(1 + x^2)$ this equation becomes

$$\frac{x\,dx}{1 + x^2} + \frac{y\,dy}{1 + y^2} = 0.$$

Integration gives

$$\tfrac{1}{2}\log(1 + x^2) + \tfrac{1}{2}\log(1 + y^2) = C.$$

This reduces to

$$(1 + x^2)(1 + y^2) = e^{2C} = C_1,$$

or

$$y^2 = \frac{C_1}{1 + x^2} - 1.$$

Illustration 2.

$$\sqrt{1 - y^2}\, dx + \sqrt{1 - x^2}\, dy = 0.$$

Then

$$\frac{dx}{\sqrt{1 - x^2}} + \frac{dy}{\sqrt{1 - y^2}} = 0,$$

and the variables are separated. Integration gives

$$\sin^{-1} x + \sin^{-1} y = C.$$

Take the sine of each member, observing that the first member is the sum of two angles, and obtain

$$x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = \sin C = C_1.$$

Exercises

Solve the following differential equations:

1. $(1 - x)dy - (1 + y)dx = 0.$ *Ans.* $(1 + y)(1 - x) = C.$

2. $\sin x \cos y\, dx = \cos x \sin y\, dy.$

3. $(x - \sqrt{1 + x^2})\sqrt{1 + y^2}\, dx = (1 + x^2)dy.$

4. $\dfrac{dy}{dx} = 5y^2x.$

5. $\dfrac{dy}{dx} + \dfrac{y^2 + 4y + 5}{x^2 + 4x + 5} = 0.$

6. $(1 + x)dy = y(1 - y)dx.$ *Ans.* $y = c(1 + x)(1 - y).$

7. $(1 - x)y\, dx + (1 + y)x\, dy = 0.$

8. $\dfrac{dy}{dx} + e^x y = e^x y^2.$

9. $(x^2 + yx^2)dy - (y^2 - xy^2)dx = 0.$

10. $x\dfrac{dy}{dx} + 2y = xy\dfrac{dy}{dx}.$

11. $3e^x \sin y\, dx + (1 - e^x) \cos y\, dy = 0.$

12. $(xy + x^3y)dy - (1 + y^2)dx = 0.$ *Ans.* $(1 + x^2)(1 + y^2) = cx^2.$

184. Homogeneous Differential Equations. The differential equation

$$M\, dx + N\, dy = 0 \tag{1}$$

is said to be *homogeneous* if M and N are homogeneous functions of x and y of the same degree.

A function $f(x, y)$ of the variables x and y is said to be *homogeneous of degree* n if after the substitutions $x = \lambda x'$, $y = \lambda y'$ have been made,

$$f(x, y) = \lambda^n f(x', y').$$

Thus

$$ax^2 + bxy + cy^2$$

is homogeneous of degree 2, for, on making the substitutions indicated, it becomes

$$\lambda^2(ax'^2 + bx'y' + cy'^2).$$

The expression

$$ax^2\sqrt{x^2 + y^2} + bx^3 \tan^{-1}\left(\frac{y}{x}\right)$$

is homogeneous of degree 3, for after the substitutions indicated above, it becomes

$$\lambda^3\left[ax'^2\sqrt{x'^2 + y'^2} + bx'^3 \tan^{-1}\left(\frac{y'}{x'}\right)\right].$$

A homogeneous differential equation of the form (1) is solved by placing $y = vx$, and thus obtaining a new differential equation in which the variables v and x are separable.

Illustration.

$$(x^2 + y^2)\, dx + 3xy\, dy = 0.$$

Let

$$y = vx.$$

Then

$$dy = v\,dx + x\,dv,$$

and

$$x^2(1 + v^2)\,dx + 3vx^2(v\,dx + x\,dv) = 0.$$
$$x^2(1 + 4v^2)\,dx + 3vx^3\,dv = 0.$$

Separating the variables,

$$\frac{dx}{x} + \frac{3v\,dv}{1 + 4v^2} = 0.$$
$$\log\,[x(1 + 4v^2)^{\frac{3}{8}}] = C,$$
$$x(1 + 4v^2)^{\frac{3}{8}} = C_1.$$

On substituting $v = \dfrac{y}{x}$ we obtain as the solution of the given equation

$$x^{\frac{1}{4}}(x^2 + 4y^2)^{\frac{3}{8}} = C_1,$$

or

$$x^2(x^2 + 4y^2)^3 = C_2.$$

Exercises

Solve the following differential equations:

1. $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}.$
2. $x^2y\,dx - (x^3 + y^3)\,dy = 0.$
3. $(8y + 10x)\,dx + (5y + 7x)\,dy = 0.$
4. $(2\sqrt{xy} - x)\,dy + y\,dx = 0.$
5. $x\dfrac{dy}{dx} = y + \sqrt{x^2 + y^2}.$
6. $x\cos\dfrac{y}{x}\dfrac{dy}{dx} = y\cos\dfrac{y}{x} - x.$
7. $x\dfrac{dy}{dx} - y = \sqrt{x^2 - y^2}.$
8. $(y - x)\,dy + y\,dx = 0.$

185. Linear Differential Equations of the First Order. The equation

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

where P and Q are functions of x only, is called a *linear differential equation*. It is of the first degree in y and its derivative. Multiply the equation by

$$e^{\int P\,dx}$$

and obtain

$$e^{\int P\,dx}\left[\frac{dy}{dx} + Py\right] = e^{\int P\,dx}Q. \tag{2}$$

The left-hand member is the derivative of

$$ye^{\int P\,dx},$$

as may be confirmed by differentiating this product. The integration of (2) gives

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + C.$$

Illustration 1.

$$\frac{dy}{dx} + xy = x^3. \tag{3}$$

Here $P = x$ and $Q = x^3$. Then

$$e^{\int P\,dx} = e^{\int x\,dx} = e^{\frac{x^2}{2}}.$$

Multiply both members of (3) by $e^{\frac{x^2}{2}}$.

$$e^{\frac{x^2}{2}}\left[\frac{dy}{dx} + xy\right] = e^{\frac{x^2}{2}}\,x^3.$$

Integration gives

$$ye^{\frac{x^2}{2}} = \int e^{\frac{x^2}{2}} x^3\,dx + C$$

$$= e^{\frac{x^2}{2}} x^2 - 2e^{\frac{x^2}{2}} + C.$$

Hence

$$y = x^2 - 2 + Ce^{-\frac{x^2}{2}}.$$

Illustration 2.

$$\frac{dy}{dx} + \frac{1}{x}y = x^2 + 3x + 4. \tag{4}$$

$$e^{\int P\,dx} = e^{\int \frac{dx}{x}} = e^{\log x} = x.$$

Multiply both members of (4) by x.

$$x\left[\frac{dy}{dx} + \frac{1}{x}y\right] = x^3 + 3x^2 + 4x.$$

Integration gives

$$xy = \frac{x^4}{4} + x^3 + 2x^2 + C,$$

or

$$y = \frac{x^3}{4} + x^2 + 2x + \frac{C}{x}.$$

This illustration is inserted to call attention to the well-known simple relation $e^{\log x} = x$, which there will be frequent occasion to use in solving equations of this type. It should be recalled that $e^{n \log x} = e^{\log (x^n)} = x^n$. Thus

$$e^{-\log x} = \frac{1}{x}.$$

Exercises

1. $\dfrac{dy}{dx} + 2xy = e^{-x^2}.$
2. $\dfrac{dy}{dx} + y \cos x = \sin 2x.$

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3. $\cos^2 x \dfrac{dy}{dx} + y = \tan x.$

4. $(x^2 + 1)\dfrac{dy}{dx} + 2xy = 4x^2.$

5. $\dfrac{dy}{dx} + \dfrac{2y}{x + 1} = (x + 1)^3.$

6. $x(1 - x^2)\,dy + (2x^2 - 1)y\,dx = ax^3\,dx.$

7. $\dfrac{dy}{dx} - n\dfrac{y}{x} = e^x x^n.$

8. $(1 + x^2)\,dy + \left(xy - \dfrac{1}{x}\right)dx = 0.$

9. $\dfrac{dy}{dx} + \dfrac{1 - 2x}{x^2}y = 1.$

10. $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy.$

186. Extended Form of the Linear Differential Equation. An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \tag{1}$$

is easily reduced to the linear form, for, on dividing (1) by y^n, we obtain

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q. \tag{2}$$

The first term of the left-hand member of (2) is, apart from a constant factor, the derivative of y^{-n+1}, which occurs in the second term. If we let $z = y^{-n+1}$, we obtain the linear differential equation

$$\frac{1}{1 - n}\frac{dz}{dx} + Pz = Q,$$

or

$$\frac{dz}{dx} - (n - 1)Pz = -(n - 1)Q.$$

Illustration 1.

$$\frac{dy}{dx} + y\cos x = y^4 \sin 2x.$$

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Dividing by y^4,

$$y^{-4}\frac{dy}{dx} + y^{-3}\cos x = \sin 2x.$$

Let $z = y^{-3}$. Then

$$y^{-4}\frac{dy}{dx} = -\tfrac{1}{3}\frac{dz}{dx}$$

and the equation becomes

$$-\tfrac{1}{3}\frac{dz}{dx} + z\cos x = \sin 2x,$$

or

$$\frac{dz}{dx} - 3z\cos x = -3\sin 2x.$$

This equation can be readily solved by **§185**; y^{-3} is to be substituted for z in the result.

Exercises

1. $\dfrac{dy}{dx} + \dfrac{1}{x}y = x^2y^6.$

2. $\dfrac{dy}{dx} + y = xy^3.$

3. $3y^2\dfrac{dy}{dx} - 7y^3 = x + 1.$

4. $(1 - x^2)\dfrac{dy}{dx} - xy = axy^2.$

5. $\dfrac{dy}{dx} + \dfrac{2}{x}y = 3x^2y^{\frac{4}{3}}.$

6. $x\dfrac{dy}{dx} + y = y^2\log x.$

7. $\dfrac{dy}{dx} + \dfrac{2}{x+1}y = \dfrac{x^3}{y^2}.$

187. Applications. Let there be an electric circuit whose resistance is R, whose coefficient of self-induction is L, and which contains an electromotive force, which at first we shall suppose constant and equal to E. It is required to find the current i at any time t after the time $t = 0$, at which the circuit was closed. The equation connecting the quantities involved is readily set up. The applied electromotive force, E, must overcome the resistance of the circuit and its self-induction. The former requires an electromotive force equal to iR, and the latter an electromotive

force proportional to the time rate of change of current, *viz.*, $\frac{di}{dt}$, and equal to $L\frac{di}{dt}$. The applied electromotive force, E, must equal the sum of these two electromotive forces.
Hence

$$L\frac{di}{dt} + Ri = E. \tag{1}$$

The student will show that, if i equals zero when t equals zero, the solution of this linear equation is

$$i = \frac{E}{R}\left[1 - e^{-\frac{R}{L}t}\right]. \tag{2}$$

If the battery or other source of electromotive force is suddenly cut out of the circuit, the current falls off in such a way that the differential equation

$$L\frac{di}{dt} + Ri = 0 \tag{3}$$

is satisfied. Show that the law according to which the current falls off is

$$i = i_0 e^{-\frac{R}{L}(t-t_0)}, \tag{4}$$

if the instant at which the battery is cut out is the time $t = t_0$ and if the current at this instant is $i = i_0$.

If the electromotive force is variable, the relation between the quantities involved in the circuit is still governed by (1),

$$L\frac{di}{dt} + Ri = E, \tag{1}$$

in which E is now variable. Suppose $E = E_0 \sin \omega t$. This supposes that an alternating electromotive force is acting in the circuit. The differential equation to be solved is

$$L\frac{di}{dt} + Ri = E_0 \sin \omega t. \tag{5}$$

Show that (see §107)

$$ie^{\frac{R}{L}t} = \frac{E_0}{L}\frac{1}{\frac{R^2}{L^2}+\omega^2}\left[\frac{R}{L}\sin\omega t - \omega\cos\omega t\right]e^{\frac{R}{L}t} + C$$

$$= \frac{E_0}{R^2+\omega^2L^2}(R\sin\omega t - \omega L\cos\omega t)e^{\frac{R}{L}t} + C$$

$$= \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t - \phi)e^{\frac{R}{L}t} + C,$$

where

$$\sin\phi = \frac{\omega L}{\sqrt{R^2+\omega^2L^2}},$$

$$\cos\phi = \frac{R}{\sqrt{R^2+\omega^2L^2}}.$$

Then

$$i = \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t - \phi) + Ce^{-\frac{R}{L}t}.$$

Since the last term becomes negligible after a short time because of the factor

$$e^{-\frac{R}{L}t},$$

it is scarcely necessary to determine C. On dropping out the last term as unimportant except in the immediate vicinity of $t = 0$, we have

$$i = \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t - \phi). \tag{6}$$

The current, therefore, alternates with the same frequency as the electromotive force, but lags behind it and differs from it in phase by ϕ. It is to be noted that the maximum value of the current is not $\frac{E_0}{R}$ but $\frac{E_0}{\sqrt{R^2+\omega^2L^2}}$. The quantity $\sqrt{R^2+\omega^2L^2}$ replaces,

in alternating currents, the resistance R of the ordinary circuit. It is called the impedance of the circuit.

188. Linear Differential Equations of Higher Order with Constant Coefficients and Second Member Zero. A typical differential equation of this class is the following:

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0, \quad (1)$$

where $a_0, a_1, \ldots, a_n$ are constants. As the equations of this class which occur in the applications are usually of the second order, we shall confine our discussion in this article to linear differential equations of the second order. Consider

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0. \quad (2)$$

Let us assume that

$$y = e^{mx} \quad (3)$$

and find, if possible, the values of m for which (3) is a solution of (2). The substitution of (3) in (2) gives

$$e^{mx}(a_0m^2 + a_1m + a_2) = 0. \quad (4)$$

The first factor cannot vanish. The second, equated to zero, gives a quadratic equation in m. Call its roots m_1 and m_2. Then (3) is a solution of (2) if m has either of the values m_1 or m_2, the roots of

$$a_0m^2 + a_1m + a_2 = 0. \quad (5)$$

The equation (5) in m, obtained from the given differential equation by writing m^2 for $\frac{d^2y}{dx^2}$ and m for $\frac{dy}{dx}$ is called the *auxiliary equation.*

Two solutions of (2) are

$$y = e^{m_1x} \quad \text{and} \quad y = e^{m_2x}.$$

Furthermore,

$$y = C_1e^{m_1x}.$$

is a solution of (2), for, after the substitution of this value of y in (2), C_1 can be taken out as a common factor and the other factor vanishes in accordance with (4) or (5). In the same way

$$y = C_2e^{m_2x}$$

is a solution of (2). And finally the sum of the two solutions

$$y = C_1e^{m_1x} + C_2e^{m_2x} \tag{6}$$

is a solution of (2). This can be seen by substituting in (2) and recalling that m_1 and m_2 are roots of (5). When m_1 is not equal to m_2, (6) is known as the *general solution* of the differential equation (2). It contains two arbitrary constants, the number which the general solution of a differential equation of the second order must contain.

The values of these constants are determined in a particular problem by two suitable conditions.

Illustration.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

The auxiliary equation is

$$m^2 - 5m + 6 = 0,$$
$$(m - 2)(m - 3) = 0.$$

Hence $m_1 = 2$, $m_2 = 3$. The general solution is then

$$y = C_1e^{2x} + C_2e^{3x}.$$

Exercises

1. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0.$
2. $\dfrac{d^2y}{dx^2} - 4y = 0.$
3. $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 10y = 0.$
4. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 12y = 0.$
5. $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} = 0.$

189. Auxiliary Equation with Equal Roots. The method just given fails when the auxiliary equation has equal roots, $m_1 = m_2$, for equation (6), §**188**, becomes

$$\begin{aligned} y &= C_1e^{m_2x} + C_2e^{m_2x} \\ &= (C_1 + C_2)e^{m_2x}. \end{aligned}$$

But $C_1 + C_2$ is an arbitrary constant and the solution contains only one arbitrary constant instead of two. When the auxiliary equation has equal roots, $m_1 = m_2$, equation (2), §**188**, can be written in the form

$$\frac{d^2y}{dx^2} - 2m_1\frac{dy}{dx} + m_1^2y = 0.$$

Its general solution is

$$y = (C_1 + C_2x)e^{m_1x}.$$

This solution can be verified by direct substitution.

Illustration.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$$

The auxiliary equation is

$$\begin{aligned} m^2 - 4m + 4 &= 0. \\ m_1 = m_2 &= 2. \end{aligned}$$

The general solution is

$$y = (C_1 + C_2x)e^{2x}.$$

Exercises

1. $4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 0.$

2. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0.$

3. $\frac{d^2y}{dx^2} = 0.$

190. Auxiliary Equation with Complex Roots. If the auxiliary equation has complex roots, the general solution can be written

in a form different from (6), §**188**. The importance of the result will be evident at once when it is observed that it contains the harmonic functions sine and cosine. If the coefficients of the given differential equation (2), §**188**, are real, and if m_1 and m_2 are complex, they must be conjugate complex numbers. Let $m_1 = a + ib$. Then $m_2 = a - ib$. Then (6) becomes

$$\begin{aligned} y &= C_1 e^{ax+ibx} + C_2 e^{ax-ibx} \\ &= e^{ax}(C_1 e^{ibx} + C_2 e^{-ibx}). \end{aligned}$$

Now, by (4) and (5), §**173**,

$$\begin{aligned} e^{ibx} &= \cos bx + i \sin bx \\ e^{-ibx} &= \cos bx - i \sin bx. \end{aligned}$$

Then

$$y = e^{ax}[(C_1 + C_2) \cos bx + i(C_1 - C_2) \sin bx].$$

On placing $C_1 + C_2 = A$ and $i(C_1 - C_2) = B$, we obtain

$$\begin{aligned} y &= e^{ax}(A \cos bx + B \sin bx) \\ &= e^{ax}\, C \cos (bx - \phi). \end{aligned}$$

In the last form the two arbitrary constants of integration are C and ϕ.

Illustration 1.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0.$$

The auxiliary equation is

$$m^2 + 4m + 13 = 0.$$

Hence

$$m = -2 \pm 3i.$$

Then

$$\begin{aligned} y &= e^{-2x}(A \cos 3x + B \sin 3x) \\ &= Ce^{-2x} \sin (3x - \phi). \end{aligned}$$

Illustration 2.

$$\frac{d^2y}{dx^2} + 4y = 0,$$
$$m^2 + 4 = 0,$$

whence

$$m = \pm 2i = 0 \pm 2i.$$

Then

$$y = A \cos 2x + B \sin 2x$$
$$= C \sin (2x - \phi).$$

Exercises

1. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0.$

2. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 13y = 0.$

3. $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 41y = 0.$

4. $\dfrac{d^2y}{dx^2} + 9y = 0.$

5. $\dfrac{d^2y}{dx^2} + y = 0.$

6. $\dfrac{d^2\theta}{dt^2} + \omega^2\theta = 0.$

191. Damped Harmonic Motion. The resistance offered by the air to the motion of a body through it is roughly proportional to the velocity, if the velocity is a moderate one. In §87, the differential equation of the motion of the simple pendulum was derived on the assumption that the force of gravity was the only force acting upon the bob of the pendulum. If the resistance of the air is also taken into account, we shall have to add to the second member of the equation, $l\dfrac{d^2\theta}{dt^2} = -g \sin \theta$, a term, $-2kl\dfrac{d\theta}{dt}$, proportional to the velocity $l\dfrac{d\theta}{dt}$ (see equation (1), §87). The differential equation of the motion is, then,

$$l\frac{d^2\theta}{dt^2} = -g \sin \theta - 2kl\frac{d\theta}{dt}. \tag{1}$$

The negative sign is used before the last term because the force due to the resistance of the air acts in a direction opposite to that

of the motion. The advantage of choosing $2k$ as the proportionality factor instead of k will appear later. k is a positive constant.

From (1) we obtain

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0. \tag{2}$$

As in §87 assume that θ is small and replace $\sin\theta$ by θ. Also let $\frac{g}{l} = \omega^2$. Then (2) becomes

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \omega^2\theta = 0. \tag{3}$$

This is a linear differential equation of the second order with constant coefficients and can be solved by the method of **§190.** The auxiliary equation is

$$m^2 + 2km + \omega^2 = 0,$$

whence

$$m = -k \pm \sqrt{k^2 - \omega^2}.$$

In the case of an ordinary pendulum in air, k is very small and much less than ω, and the expression under the radical sign is negative. We write then

$$m = -k \pm i\sqrt{\omega^2 - k^2},$$

$\omega^2 - k^2$ being positive.

The solution of (3) is

$$\theta = Ae^{-kt}\cos[t\sqrt{\omega^2 - k^2} - \epsilon],$$

or, multiplying both sides by l and replacing Al by B,

$$s = Be^{-kt}\cos[t\sqrt{\omega^2 - k^2} - \epsilon]. \tag{4}$$

The motion is a damped harmonic motion. The amplitude decreases with the time. The period $\frac{2\pi}{\sqrt{\omega^2 - k^2}}$ is a little greater than $\frac{2\pi}{\omega}$, the period of the free motion.

Since k is very small in comparison with ω, we can, for an approximate solution of our problem, neglect k^2 in comparison with ω^2. Equation (4) becomes

$$s = Be^{-kt} \cos (\omega t - \epsilon). \tag{5}$$

This represents the motion with a high degree of approximation.

The arbitrary constants B and ϵ can be determined by suitable initial conditions. For example, let it be given that $s = s_0$ and $\frac{ds}{dt} = 0$ when $t = 0$. On differentiating (5) we obtain

$$\frac{ds}{dt} = Be^{-kt}[-k \cos (\omega t - \epsilon) - \omega \sin (\omega t - \epsilon)]. \tag{6}$$

For $t = 0$ we obtain from (5) and (6)

$$s_0 = B \cos \epsilon$$
$$0 = B(-k \cos \epsilon + \omega \sin \epsilon).$$

From the latter of these two equations

$$\tan \epsilon = \frac{k}{\omega}.$$

From the former

$$B = s_0 \sec \epsilon = s_0\sqrt{1 + \frac{k^2}{\omega^2}} = s_0$$

to the degree of approximation used above. We have then as the approximate equation of motion

$$s = s_0e^{-kt} \cos (\omega t - \epsilon) \tag{7}$$

where

$$\epsilon = \tan^{-1}\frac{k}{\omega} = \frac{k}{\omega}, \text{ approximately.}$$

Since k is very small, ϵ is very small.

It follows from (5) and (7) that the period of the pendulum in the case just considered is very little different from that of the same pendulum swinging in a vacuum. The amplitude of the swing, however, is affected and diminishes continually with the time.

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